

The bulk, surface and corner free energies of the square lattice Ising model

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We use Kaufman's spinor method to calculate the bulk, surface and corner free energies f_b, f_s, f'_s, f_c of the anisotropic square lattice zero-field Ising model for the ordered ferromagnetic case. For f_b, f_s, f'_s our results of course agree with the early work of Onsager, McCoy and Wu. We also find agreement with the conjectures made by Vernier and Jacobsen (VJ) for the isotropic case. We note that the corner free energy f_c depends only on the elliptic modulus k that enters the working, and not on the argument v , which means that VJ's conjecture applies for the full anisotropic model. The only aspect of this paper that is new is the actual derivation of f_c , but by reporting all four free energies together we can see interesting structures linking them.

KEY WORDS: Statistical mechanics, lattice models, exactly solved models, surface and corner free energies

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1. Introduction

Vernier and Jacobsen[1] considered a number of two-dimensional lattice models in statistical mechanics that are “exactly solved” in the sense that their bulk free energies (and where appropriate their order parameters) have been calculated exactly. For most of them their surface free energies have not been calculated, and for none of them their corner free energies.

If Z is the partition function of a square lattice of M rows and N columns, then we expect on physical grounds that when M and N are large, taking $\beta = 1/k_B T$ where k_B is Boltzmann’s constant and T the temperature, that for a non-critical system ($T \neq T_c$)

$$\beta^{-1} \log Z = -MNf_b - Mf_s - Nf'_s - f_c + O(e^{-\gamma M}, e^{-\gamma' N}), \quad (1.1)$$

where f_b, f_s, f'_s and f_c are the bulk, vertical surface, horizontal surface and corner free energies, respectively, and γ, γ' each have positive real part.

The “partition function per site” κ is

$$\kappa = e^{-\beta f_b} = \lim_{M, N \rightarrow \infty} Z^{1/MN}$$

and for solvable models can usually be written as a product over m of factors such as $1 \pm wq^m$ or $(1 \pm wq^m)^m$, where q, z are parameters that occur naturally in the mathematical calculation. Vernier and Jacobsen considered only the

isotropic cases of these models, when z is some power of q , and obtained quite long series expansions for $e^{-\beta f_s} = e^{-\beta f'_s}$ and $e^{-\beta f_c}$. They looked for, and usually found, a simple repeat pattern in the product expansion. This enabled them to conjecture results for the surface and corner free energies.

Baxter and Owczarek[2] calculated the surface free energy of the square lattice Potts model when the number of states per spin is Q and $Q \leq 4$, when the system is critical and the free energies are integrals rather than products. Vernier and Jacobsen considered the case $Q > 4$. In another paper[3] the author has extended his and Owczarek's working to $Q > 4$, thereby obtaining f_s, f'_s exactly. The result does indeed agree with Vernier and Jacobsen's conjecture for the isotropic case.

In this paper we calculate the bulk, surface and corner free energies of the two-dimensional Ising model on the square lattice for the ferromagnetically ordered case, when $T < T_c$.

The square lattice Ising model was the first of such models to be solved, the bulk free energy f_b being calculated by Onsager in 1944. [4] It is simpler than other solvable models because the partition function can be written as a pfaffian (the square root of an anti-symmetric determinant).[6] In 1967 McCoy and Wu[7, eqn.4.24b][8, p.126, eqn.4.24b] calculated the surface free energies f_s, f'_s . Our results for these quantities do indeed agree with theirs, as we show in Appendix A.

The partition function of the Ising model is defined in (2.1) below and contains two parameters, the vertical and horizontal interaction coefficients H and H' . In terms of the elliptic functions that we introduce, these correspond to k and v (the modulus and an argument), or equivalently to the q and w defined in (6.26).

For the isotropic case $H = H'$, so v and w are fixed, as in (6.38) ($v = iK'/2$, $w = q^{1/4}$). Our results for f_b, f_s, f'_s, f_c do indeed agree with Vernier and Jacobsen's conjectures.

It is known that there is a simple "inversion relation" method which can usually be used to obtain the bulk free energy of a solved model (i.e. one which satisfies a "Yang-Baxter" relation, which means that the free energies have simple analyticity properties). Here we show in section 7 that this method can be extended to obtain f_s and f'_s , and to show that f_c is independent of the anisotropy parameter v (or w). Unlike the main derivation in this paper, this inversion relation method makes some assumptions, notably that $e^{-\beta f_b}, e^{-\beta f_s}, e^{-\beta f'_s}, e^{-\beta f_c}$ are analytic functions of w in an annulus containing the "inversion points" $w^2 = 1$ and $w^2 = q$, except possibly for known poles or zeros at those points. It is therefore not rigorous, but it provides a much easier route to the calculation of the free energies.

The self-dual Potts model has very similar properties.[3] For that and other solvable models O'Brien, Pearce, Behrend and Batchelor[14, 15, 16] have obtained surface free energies by using the reflection Yang-Baxter relations together with the inversion identities and inversion relations. Much of their work concerns a lattice which differs from ours by rotation through 45° , for which the boundaries are changed and one would expect from simple, plausible but not rigorous, arguments there to be two inversion relations. However, Pearce[17, eqn. 52] has been able to rotate their exact results through 45° to obtain

an extra relation for the surface free energy f'_s of the self-dual Potts model with the same orientation of the lattice as here. Possibly these methods could be used to provide a rigorous justification of the inversion relation method of section 7.

It is also true for the self-dual Potts model that f_c is independent of the anisotropy parameter. Thus for the rectangular lattice f_c is like the order parameters, notably the spontaneous magnetization \mathcal{M}_0 , which also are independent of the anisotropy parameter. (However, the same is probably not true for other lattices, such as the triangular.)

Our derivation does manifest an interesting property of the four free energies f_b, f_s, f'_s, f_c . Apart from additive terms that are logarithms of simple rational functions of the Boltzmann weights, they are integrals or sums of functions that can be written as $A_1 B_1, A_1 B_2, A_2 B_1, A_2 B_2$, respectively. Thus if one knows f_b, f_s, f'_s , then f_c is determined, to within such an additive term. The additive terms do not affect the singularities at the critical point, so one should be able to obtain the critical behaviour of f_c from a knowledge of f_b, f_s, f'_s . We do note in section 9 that the free energies have critical singularities of the form $(T_c - T)^{2-\alpha} \log(T_c - T)$, where $\alpha = 0, 1, 1, 2$ for f_b, f_s, f'_s, f_c , respectively. Identifying the four exponents in an obvious way, this implies that

$$\alpha_f + \alpha_c = \alpha_s + \alpha'_s, \quad (1.2)$$

and maybe this equation has more general application.

In this paper we consider only the ferromagnetically ordered case of the Ising model, when H, H' are positive and $T < T_c$. We expect the extension to the disordered case, when $T > T_c$, to be calculable in a similar manner.

We emphasize that our results are exact for the non-critical Ising model. Cardy and Peschel,[18] and Wu and Izmailyan[19] have considered the model at criticality, where one can use and compare with the predictions of conformal field theory.

There has also been much work on the surface and corner magnetizations of the Ising model,[20, 21] but we do not discuss this here.

2. The Ising model partition function

We consider the Ising model on the square lattice of M rows and N columns, as shown by the solid lines and the circles in Fig. 1. On each site i we place a “spin” σ_i , with values -1 and $+1$. The vertical and horizontal interaction coefficients are $H = \beta J$, $H' = \beta J'$ and the partition function is

$$Z = \sum_{\sigma} \exp \left(H \sum_{\langle ij \rangle} \sigma_i \sigma_j + H' \sum_{\langle ij \rangle} \sigma_i \sigma_j \right) \quad (2.1)$$

the first of the two inner sums being over all $(M-1)N$ vertical edges i, j , the second over all $M(N-1)$ horizontal edges i, j , and the outer sum over all 2^{MN} values of the MN spins.

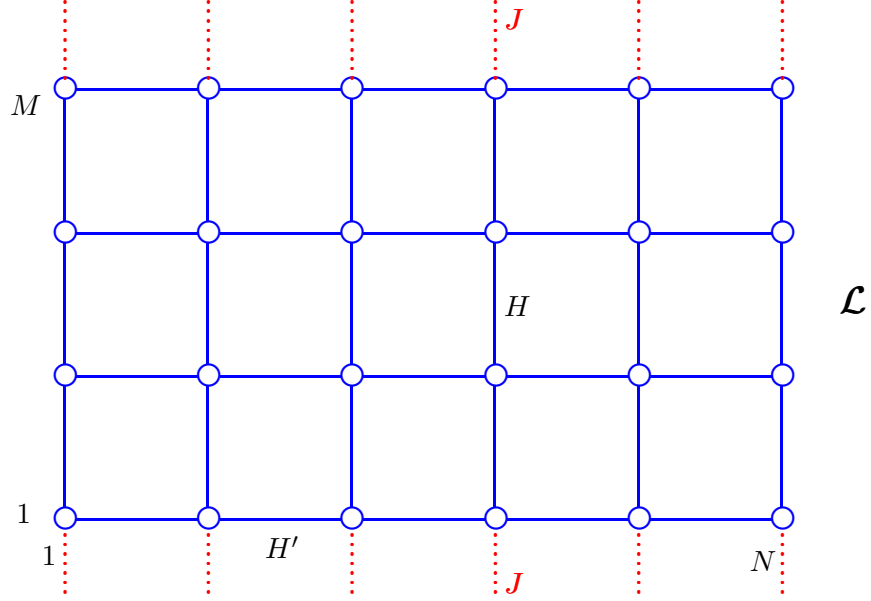


Figure 1: The square lattice \mathcal{L} (of 4 rows and 6 columns), indicating the vertical and horizontal interaction coefficients H, H' .

Let $\sigma = \{\sigma_1, \dots, \sigma_N\}$ be the spins on a row of the lattice, and $\sigma' = \{\sigma'_1, \dots, \sigma'_N\}$ be the spins on the row above. Then we can define the 2^N -dimensional row-to-row transfer matrices V_1, V_2 , with elements

$$(V_1)_{\sigma, \sigma'} = \prod_{i=1}^N e^{H \sigma_i \sigma'_i}$$

$$(V_2)_{\sigma, \sigma'} = \exp[H' \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1}] \prod_{i=1}^N \delta_{\sigma_i, \sigma'_i} \quad (2.2)$$

The matrix V_2 is diagonal.

Let

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2.3)$$

and define s_j, c_j to be the 2^N -dimensional matrices

$$s_j = \mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes \mathbf{s} \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1},$$

$$c_j = \mathbf{1} \otimes \dots \otimes \mathbf{1} \otimes \mathbf{c} \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}, \quad (2.4)$$

\mathbf{s}, \mathbf{c} on the RHS being in position j . Then

$$V_1 = (2 \sinh 2H)^{N/2} \exp\{H^* \sum_{i=1}^N c_i\} \quad (2.5)$$

$$V_2 = \exp\{H' \sum_{i=1}^{N-1} s_i s_{i+1}\}, \quad (2.6)$$

where, as in Onsager,[\[4, eq. 14\]](#)

$$\tanh H^* = e^{-2H} . \quad (2.7)$$

2.1. Spinor representatives

Kaufman[\[11\]](#) simplified Onsager's calculation by using spinor (or free-fermion) operators, where one represents the 2^N -dimensional matrices V_1, V_2 by $2N$ -dimensional matrices \hat{V}_1, \hat{V}_2 . Here we use this method.

For $j = 1, \dots, N$, define

$$\Gamma_j = c_1 c_2 \cdots c_{j-1} s_j, \quad \Gamma_{j+N} = i \Gamma_j c_j \quad (2.8)$$

and note that

$$c_j = -i \Gamma_j \Gamma_{j+N}, \quad s_j s_{j+1} = -i \Gamma_{j+N} \Gamma_{j+1} . \quad (2.9)$$

Then

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2 \delta_{ij} I, \quad (2.10)$$

I being the 2^N -dimensional identity matrix. Let X, Y be two matrices such that

$$X \Gamma_i X^{-1} = \sum_{j=1}^{2N} x_{j,i} \Gamma_j, \quad Y \Gamma_i Y^{-1} = \sum_{j=1}^{2N} y_{j,i} \Gamma_j \quad (2.11)$$

and set $T = XY$. Then, for $i = 1, \dots, 2N$,

$$T \Gamma_i T^{-1} = \sum_{j=1}^{2N} t_{ji} \Gamma_j, \quad (2.12)$$

where

$$t_{ij} = \sum_{m=1}^{2N} x_{im} y_{mj} . \quad (2.13)$$

It follows that such matrices form a group \mathcal{G} . Let \hat{X} be the $2N$ -dimensional matrix with elements x_{ij} , and similarly for \hat{Y}, \hat{T} , then we call $\hat{X}, \hat{Y}, \hat{T}$ the *representatives* of X, Y, T and $T = XY$ implies $\hat{T} = \hat{X} \hat{Y}$.

For arbitrary ρ_1, \dots, ρ_N , let

$$X = \exp\left\{\sum_{j=1}^N \rho_j c_j\right\} = \exp\left\{-i \sum_{j=1}^N \rho_j \Gamma_j \Gamma_{j+N}\right\}, \quad (2.14)$$

then, using (2.10),

$$X \Gamma_j X^{-1} = \cosh 2\rho_j \Gamma_j + i \sinh 2\rho_j \Gamma_{j+N}$$

$$X \Gamma_{j+N} X^{-1} = -i \sinh 2\rho_j \Gamma_j + \cosh 2\rho_j \Gamma_{j+N}$$

so

$$\hat{X} = \begin{pmatrix} A & -i B \\ i B^T & C \end{pmatrix} \quad (2.15)$$

where the A, B, C are N -dimensional diagonal matrices with entries

$$A_{ij} = C_{ij} = (\cosh 2\rho_j) \delta_{ij}, \quad B_{ij} = (\sinh 2\rho_j) \delta_{ij} \quad (2.16)$$

The $2N$ eigenvalues of \widehat{X} are therefore $e^{2\rho_j}$ and $e^{-2\rho_j}$ for $j = 1, \dots, N$.

For *any* matrix X within \mathcal{G} , it follows that if the eigenvalues of its representative \widehat{X} are $e^{2\rho_j}$ and $e^{-2\rho_j}$ (for $j = 1, \dots, N$) then there must be an invertible matrix P (also within the group) such that

$$P X P^{-1} = R \exp\left\{\sum_{i=1}^N \rho_i c_i\right\}. \quad (2.17)$$

Thus the eigenvalues of X are

$$R e^{\pm\rho_1 \pm \rho_2 \pm \dots \pm \rho_N} \quad (2.18)$$

for all 2^N choices of the signs, and the scalar factor R can be determined from

$$2^N \log R = \log \det X = \text{trace} \log X. \quad (2.19)$$

The trace of X is therefore

$$\text{trace } X = R \prod_{j=1}^N 2 \cosh \rho_j = R \prod_{j=1}^N (\Lambda_j + 2 + 1/\Lambda_j)^{1/2} \quad (2.20)$$

where the $\Lambda_j = e^{2\rho_j}$ and the eigenvalues of \widehat{X} are $\Lambda_1, \dots, \Lambda_N, \Lambda_1^{-1}, \dots, \Lambda_N^{-1}$.

From (2.9), the matrices V_1, V_2 are exponentials of quadratic forms in the Γ_j :

$$V_1 = (2 \sinh 2H)^{N/2} \exp\left\{-i H^* \sum_{i=1}^N \Gamma_i \Gamma_{i+N}\right\} \quad (2.21)$$

$$V_2 = \exp\left\{-i H' \sum_{i=1}^{N-1} \Gamma_{i+N} \Gamma_{i+1}\right\}. \quad (2.22)$$

Then, for $1 \leq j \leq N$,

$$V_1 \Gamma_j V_1^{-1} = \cosh 2H^* \Gamma_j + i \sinh 2H^* \Gamma_{j+N}$$

$$V_1 \Gamma_{j+N} V_1^{-1} = -i \sinh 2H^* \Gamma_j + \cosh 2H^* \Gamma_{j+N}$$

and for $1 \leq j \leq N-1$,

$$V_2 \Gamma_{j+N} V_2^{-1} = \cosh 2H' \Gamma_{j+N} + i \sinh 2H' \Gamma_{j+1}$$

$$V_2 \Gamma_{j+1} V_2^{-1} = -i \sinh 2H' \Gamma_{j+N} + \cosh 2H' \Gamma_{j+1}.$$

Since V_2 commutes with Γ_1 and Γ_{2N} ,

$$V_2 \Gamma_1 V_2^{-1} = \Gamma_1, \quad V_2 \Gamma_{2N} V_2^{-1} = \Gamma_{2N}.$$

It follows that V_1, V_2 , belong to \mathcal{G} and have representatives $\widehat{V}_1, \widehat{V}_2$ of the form (2.15). Set

$$c^* = \cosh 2H^*, \quad s^* = \sinh 2H^*, \quad c' = \cosh 2H', \quad s' = \sinh 2H', \quad (2.23)$$

then for V_1 ,

$$A = C = c^* \mathbf{1}, \quad B = s^* \mathbf{1} \quad (2.24)$$

where $\mathbf{1}$ is now the identity N by N matrix. For V_2 ,

$$A = \begin{pmatrix} 1 & 0 & .. & 0 & 0 \\ 0 & c' & .. & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & c' & 0 \\ 0 & 0 & .. & 0 & c' \end{pmatrix}, \quad C = \begin{pmatrix} c' & 0 & .. & 0 & 0 \\ 0 & c' & .. & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & c' & 0 \\ 0 & 0 & .. & 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & .. & 0 & 0 & 0 \\ -s' & 0 & .. & 0 & 0 & 0 \\ 0 & -s' & .. & 0 & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & .. & -s' & 0 & 0 \\ 0 & 0 & .. & 0 & -s' & 0 \end{pmatrix} \quad (2.25)$$

so for V_2 the N by N matrices A, C are diagonal and B is one-off diagonal.

2.2. The top-to-bottom boundary condition

We handle the top and bottom boundary conditions as follows. We link the top and bottom rows by extra vertical edges, shown as dotted (red) lines in Figure 1, and allow the top and bottom spins to interact with an interaction coefficient J . This changes the boundary conditions to the familiar cylindrical ones, but we can readily regain the original open boundary conditions by taking the limit $J \rightarrow 0$. Let W be the transfer matrix of this row, given by V_1 , but with H replaced by J . Then the partition function is

$$Z = \text{trace}(WV_2V_1V_2 \cdots V_1V_2), \quad (2.26)$$

there being M factors V_2 , and $M - 1$ factors V_1 , in the matrix product. We note that

$$WV_2V_1V_2 \cdots V_1V_2 = F \times \text{product of traceless matrices} \quad (2.27)$$

where

$$F = (2 \sinh 2J)^{N/2} (2 \sinh 2H)^{N(M-1)/2}. \quad (2.28)$$

The matrix W , like V_1 and V_2 , belongs to the group \mathcal{G} . Let

$$\widehat{U} = \widehat{V}_2 \widehat{V}_1 \widehat{V}_2 \cdots \widehat{V}_2 \quad (2.29)$$

Then the representative of $WV_2V_1V_2 \cdots V_1V_2$ is

$$\widehat{W} \widehat{U}.$$

From (2.19) and (2.20), it follows that

$$Z = F \prod_{j=1}^N (\Lambda_j + 2 + 1/\Lambda_j)^{1/2}, \quad (2.30)$$

where $\Lambda_1, \dots, \Lambda_N, \Lambda_1^{-1}, \dots, \Lambda_N^{-1}$ are the $2N$ eigenvalues of $\widehat{W}\widehat{U}$.

Now we take the limit $J \rightarrow 0$ to regain the desired closed boundary conditions. Analogously to (2.7),

$$\tanh J^* = e^{-2J}, \quad (2.31)$$

so $J^* \rightarrow \infty$. Then all the non-zero elements of $\widehat{W} \rightarrow \infty$ and

$$\lim_{J \rightarrow 0} 2J \widehat{W} = P = \begin{pmatrix} I & -iI \\ iI & I \end{pmatrix}. \quad (2.32)$$

This matrix P is of rank N . It follows that we can choose the $\Lambda_1, \dots, \Lambda_N$ to be of order $1/J$, tending to ∞ , while the other N eigenvalues of $\widehat{W}\widehat{U}$ are of order J , tending to zero.

Hence (2.30) becomes

$$Z = F (\Lambda_1 \cdots \Lambda_N)^{1/2} \quad (2.33)$$

and now $2J\Lambda_1, \dots, 2J\Lambda_N$ are the non-zero eigenvalues of $P\widehat{U}$, which in turn are the eigenvalues of the N by N matrix

$$Q = (I - iI) \widehat{V}_2 \widehat{V}_1 \widehat{V}_2 \cdots \widehat{V}_2 \begin{pmatrix} I \\ iI \end{pmatrix}. \quad (2.34)$$

From (2.28) and (2.29), it follows that

$$Z = 2^{N/2} (2 \sinh 2H)^{N(M-1)/2} (\det Q)^{1/2} \quad (2.35)$$

so we have reduced the problem to one of calculating an N by N determinant.

3. Calculation of $\det Q$

Eqn. (2.34) can be written

$$Q = (I - iI) (\widehat{V}_2 \widehat{V}_1)^{M-1} \widehat{V}_2 \begin{pmatrix} I \\ iI \end{pmatrix}, \quad (3.1)$$

which leads us to look for the eigenvalues λ of $\widehat{V}_2 \widehat{V}_1$. Let \mathbf{y} be one such eigenvector, and \mathbf{x} a vector related to it by the eigenvalue equations

$$\lambda \mathbf{y} = \widehat{V}_1 \mathbf{x}, \quad \mathbf{x} = \widehat{V}_2 \mathbf{y}. \quad (3.2)$$

These \mathbf{x}, \mathbf{y} are of dimension $2N$, we can write the equations more explicitly if we define N -dimensional vectors x, x', y, y' so that

$$\mathbf{x} = \begin{pmatrix} x \\ x' \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad (3.3)$$

Then the eigenvalue equations (3.2) are

$$\begin{aligned} \lambda y_j &= c^* x_j - i s^* x'_j, \quad \lambda y'_j = i s^* x_j + c^* x'_j, \\ x_j &= c' y_j + i s' y'_{j-1}, \quad x'_j = -i s' y_{j+1} + c' y'_j, \end{aligned} \quad (3.4)$$

where $j = 1, \dots, N$, except that $j = 1$ is excluded in the third equation, and $j = N$ in the fourth. Instead, the corresponding equations are

$$x_1 = y_1, \quad x'_N = y'_N. \quad (3.5)$$

We first look for a solution of the form

$$\begin{aligned} x_j &= A z^{j-1}, \quad y_j = B z^{j-1} \\ x'_j &= A' z^{j-1}, \quad y'_j = B' z^{j-1} \end{aligned} \quad (3.6)$$

and find it works for the equations (3.4) provided that

$$\begin{aligned} \lambda B &= c^* A - i s^* A', \quad \lambda B' = i s^* A + c^* A' \\ A &= c' B + i s' B' / z, \quad A' = -i s' z B + c' B'. \end{aligned} \quad (3.7)$$

These are four homogeneous linear equations in A, A', B, B' . The determinant must be zero, which gives

$$\lambda^2 - [2c'c^* - s's^*(z + z^{-1})]\lambda + 1 = 0. \quad (3.8)$$

This equation is unchanged by replacing z by z^{-1} , so if (3.4) is one possible ansatz, another one for the same value of the eigenvalue λ is obtained by inverting z .

The eqns. (3.7) are unchanged by inverting z and simultaneously replacing A, A', B, B' by $-A', A, -B', B$; also, if λ is fixed, the equations (3.4) are linear. The more general ansatz

$$\begin{aligned} x_j &= A z^{j-1} - g A' z^{1-j}, \quad y_j = B z^{j-1} - g B' z^{1-j} \\ x'_j &= A' z^{j-1} + g A z^{1-j}, \quad y'_j = B' z^{j-1} + g B z^{1-j} \end{aligned} \quad (3.9)$$

(for $j = 1, \dots, N$) therefore satisfies equations (3.4) for arbitrary g , provided only that (3.7) holds.

We introduce a parameter α such that

$$g = i z^{N-1} \alpha. \quad (3.10)$$

Then (3.9) becomes

$$\begin{aligned} x_j &= A z^{j-1} - i \alpha A' z^{N-j}, \quad y_j = B z^{j-1} - i \alpha B' z^{N-j} \\ x'_j &= A' z^{j-1} + i \alpha A z^{N-j}, \quad y'_j = B' z^{j-1} + i \alpha B z^{N-j} \end{aligned} \quad (3.11)$$

We now attempt to satisfy the boundary conditions (3.5). It is convenient to define y'_0 by the third of the equations (3.4) (with $j = 1$), and y_{N+1} by the fourth (with $j = N$). Then we can write the boundary conditions as

$$(c' - 1)y_1 = -i s' y'_0, \quad (c' - 1)y'_N = i s' y_{N+1}, \quad (3.12)$$

These y'_0 and y_{N+1} will also be given by (3.9), so we obtain

$$\begin{aligned} (c' - 1)Bz + i s' B' &= iz^N \alpha [(c' - 1)B' - i s' z B] \\ (c' - 1)Bz + i s' B' &= iz^N \alpha^{-1} [(c' - 1)B' - i s' z B] \end{aligned} \quad (3.13)$$

We take

$$\alpha^2 = 1, \quad (3.14)$$

i.e. $\alpha = +1$ or -1 , to ensure that these equations are the same. That we can do this is a reflection of the fact that \widehat{V}_1 and \widehat{V}_2 are both invariant under replacing rows and columns $1, \dots, 2N$ by $2N, \dots, 1$, and negating rows and columns $N + 1, \dots, 2N$.

The c', s' are related to one another, as are c^*, s^* . We express them in terms of

$$u = \tanh H', \quad t = \tanh H^* = e^{-2H} \quad (3.15)$$

as

$$c' = \frac{1 + u^2}{1 - u^2}, \quad s' = \frac{2u}{1 - u^2}, \quad c^* = \frac{1 + t^2}{1 - t^2}, \quad s^* = \frac{2t}{1 - t^2}. \quad (3.16)$$

Then

$$e^{-2H} = t, \quad e^{-2H'} = \frac{1 - u}{1 + u}, \quad \sinh 2H = \frac{1 - t^2}{2t}. \quad (3.17)$$

Here we consider the ordered ferromagnetic phase of the Ising model, when H, H', H^* are all positive and

$$\sinh 2H \sinh 2H' > 1. \quad (3.18)$$

Then t, u are real and

$$0 < t < u < 1. \quad (3.19)$$

The eqns. (3.13) both become

$$B' = izB(u - \alpha z^N)/(1 - \alpha u z^N). \quad (3.20)$$

We can eliminate λ, A, A' between the equations (3.7) to obtain

$$[(u + u^{-1})z - (t + t^{-1})z^2]B^2 + 2i(1 - z^2)BB' + [(u + u^{-1})z - t - t^{-1}]B'^2 = 0$$

Making the substitution (3.20), we find that α only enters the numerator via α^2 . Using (3.14), we obtain $P(z) = 0$, where

$$\begin{aligned} P(z) &= z^{2N}(z - tu)(z - u/t) - (1 - tuz)(1 - uz/t) \\ &= z^{2N+2} - c_1 z^{2N+1} + \dots + c_1 z - 1 \end{aligned} \quad (3.21)$$

where $c_1 = u(t + 1/t)$.

This $P(z)$ is a polynomial in z , of degree $2N + 2$. Its zeros are

$$1, -1, z_1, z_2, \dots, z_N, z_N^{-1}, z_{N-1}^{-1}, \dots, z_1^{-1} \quad (3.22)$$

Substituting $z = 1$ or $z = -1$ into the above equations for \mathbf{x}, \mathbf{y} gives $\mathbf{x} = \mathbf{y} = \mathbf{0}$, so these solutions are spurious and we ignore them. As indicated in Figure 2, the remaining $2N$ zeros occur in inverse pairs, $2N - 2$ of them being on the unit circle, and 2 on the positive real axis. We choose z_1, \dots, z_{N-1} to lie on the unit circle, in the upper half-plane, ordered from left to right, and z_N to lie on the real axis, between 0 and 1, as indicated in Figure 2. The corresponding eigenvalues $\lambda_1, \dots, \lambda_N$ are real and positive. If we take

$$\alpha_j = (-1)^{N-j}, \quad (3.23)$$

for all j , then

$$\lambda_j > 1 \text{ for } j = 1, \dots, N; \quad \lambda_j < 1 \text{ for } j = N + 1, \dots, 2N. \quad (3.24)$$

Using (3.21),

$$\alpha_j z_j^N = \sqrt{\frac{(1 - tuz_j)(uz_j - t)}{(z_j - tu)(u - tz_j)}} \quad (3.25)$$

where the square root should be taken to be in the right-half of the complex plane for $1 \leq j \leq N$, in the left-half plane for $N + 1 \leq j \leq 2N$. (It is straightforward to verify that these choices are consistent in the limit $t \rightarrow 0$. By continuity it follows that they are consistent for all u, t satisfying (3.19).)

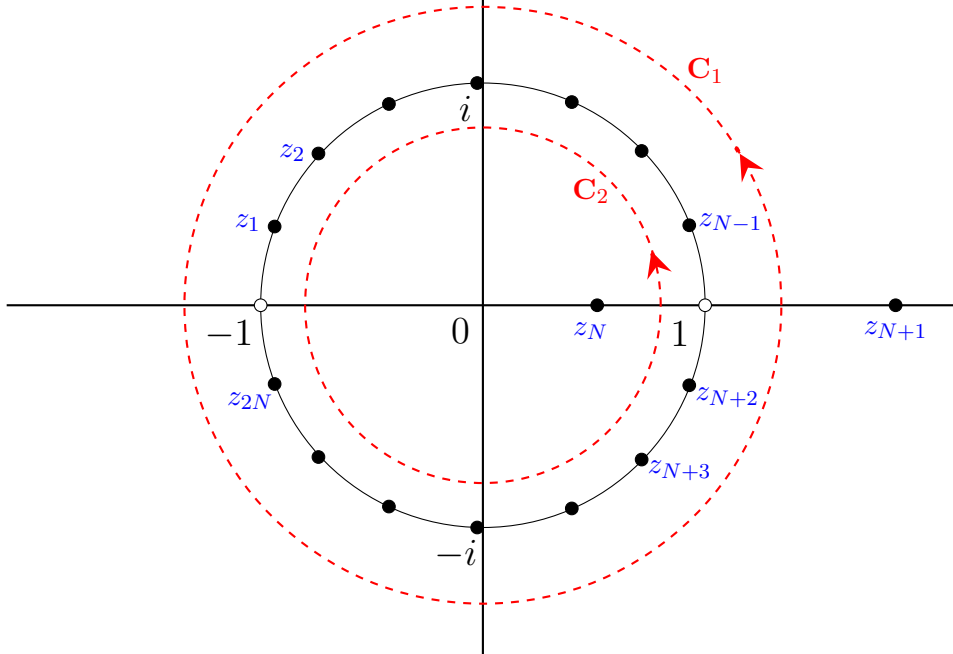


Figure 2: The positions and ordering of the zeros of (3.21); $z_j z_{2N+1-j} = 1$ for all j .

For N large, z_N is close to t/u and z_{N+1} is close to u/t :

$$z_N = t/u + O(t/u)^{2N}, \quad z_{N+1} = u/t + O(t/u)^{2N}, \quad (3.26)$$

in agreement with (3.21).

In particular, $\alpha_N = 1, \alpha_{N+1} = -1$. For all j ,

$$z_{2N+1-j} = z_j^{-1}, \quad \lambda_{2N+1-j} = \lambda_j^{-1}, \quad \alpha_{2N+1-j} = -\alpha_j. \quad (3.27)$$

This gives all the distinct $2N$ eigenvectors. If $z = z_m$, we write the vectors \mathbf{x}, \mathbf{y} (as given above), as $\mathbf{x}_m, \mathbf{y}_m$. Let X be the $2N$ by $2N$ matrix with columns $\mathbf{x}_1, \dots, \mathbf{x}_{2N}$. Similarly for Y , and let \mathcal{D} be the diagonal matrix with entries $\lambda_1, \dots, \lambda_{2N}$. Then (3.2), (3.1) become

$$\begin{aligned} \widehat{V}_1 X &= Y \mathcal{D}, \quad \widehat{V}_2 Y = X, \\ Q &= (I - iI) X \mathcal{D}^{M-1} Y^{-1} \begin{pmatrix} I \\ iI \end{pmatrix}. \end{aligned} \quad (3.28)$$

3.1. Explicit expressions

From (3.20), we can take

$$B = 1 - \alpha u z^N, \quad B' = iz(u - \alpha z^N) \quad (3.29)$$

Then

$$\begin{aligned} y_j &= z^{j-1} - z^{2N+1-j} - \alpha u z^N (z^{j-1} - z^{1-j}) \\ y'_j &= iu(z^j - z^{2N-j}) - i\alpha z^N (z^j - z^{-j}) \end{aligned} \quad (3.30)$$

Substituting these expressions into the RHS of the second pair of the equations (3.4), we find

$$\begin{aligned} x_j &= z^{j-1} - z^{2N+1-j} + \alpha u z^N (z^{j-1} - z^{1-j}) \\ x'_j &= -iu(z^j - z^{2N-j}) - i\alpha z^N (z^j - z^{-j}), \end{aligned} \quad (3.31)$$

so x_j, x'_j have the same forms as y_j, y'_j , but with u negated.

For the moment, regard z_1, \dots, z_N as arbitrary, but with the other z 's given by (3.27) and the $\alpha_1, \dots, \alpha_{2N}$ by (3.23), and consider the determinant of X . It will be a Laurent multinomial in z_1, \dots, z_N .

Column m of X is the column vector with entries $x_1, \dots, x_N, x'_1, \dots, x'_N$, as given above with $z = z_m, \alpha = \alpha_m$. If two even (or two odd) columns have the same value of z , then, using (3.23), the column vectors will be the same and the determinant of X will vanish.

It follows that the multinomial expression for $\det X$ contains factors of the form $z_j - z_{j+2r}$. Arguing similarly, we show in Appendix B that

$$\det Y = \epsilon_N 2^N (1-u^2)^{N-1} \prod_{j=1}^N \frac{(1-z_j^2)^2}{z_j^{2N}} \prod_{m=j+1}^N (z_j - z_m)^2 (1 - z_j z_m)^2, \quad (3.32)$$

where

$$\epsilon_N = 1 \text{ if } N \text{ is even, } \epsilon_N = -i \text{ if } N \text{ is odd.} \quad (3.33)$$

We also find that

$$X^T X = Y^T Y = \mathcal{W}S = S\mathcal{W} \quad (3.34)$$

where

$$\mathcal{W} = \begin{pmatrix} \tau_1 & 0 & \dots & 0 & 0 \\ 0 & \tau_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tau_2 & 0 \\ 0 & 0 & \dots & 0 & \tau_1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (3.35)$$

Taking square roots, choosing the sign appropriately, it follows that

$$\det Y = \epsilon_N \tau_1 \tau_2 \cdots \tau_N \quad (3.36)$$

and that

$$Q = (I - iI) X \mathcal{D}^{M-1} \mathcal{W}^{-1} S Y^T \begin{pmatrix} I \\ iI \end{pmatrix}.$$

Set

$$L = (I - iI) X, \quad R = S Y^T \begin{pmatrix} I \\ iI \end{pmatrix}, \quad (3.37)$$

so

$$Q = L \mathcal{D}^{M-1} \mathcal{W}^{-1} R. \quad (3.38)$$

From (3.3) and (3.37), the elements of L, R are

$$L_{j,m} = x_j - i x'_j, \quad R_{2N+1-m,j} = y_j + i y'_j, \quad (3.39)$$

where x_j, x'_j, y_j, y'_j are defined by (3.31), (3.30) with $z = z_m, \alpha = \alpha_m$.

Define the N by $2N$ matrix \tilde{C} by

$$\tilde{C}_{j,m} = -i(z_m^{j-1} + \alpha_m z_m^{N-j})(1 + \alpha_m u z_m^N - u z_m - \alpha_m z_m^{N+1})/z_m^N \quad (3.40)$$

Then we find that

$$L_{j,m} = i z_m^N \tilde{C}_{j,m}, \quad R_{m,j} = -i z_m^{-N} \tilde{C}_{m,j} \quad (3.41)$$

where $z = z_m, \alpha = \alpha_m$ are the values of z, α for the m th eigenvector. We have used the relations $z_{2N+1-m} = 1/z_m, \alpha_{2N+1-m} = -\alpha_m$.

Because \mathcal{D}, \mathcal{W} are diagonal, the factors $i z_m^N, -i z_m^N$ in (3.41) cancel out of (3.38), so

$$Q = \tilde{C} \mathcal{D}^{M-1} \mathcal{W}^{-1} \tilde{C}^T. \quad (3.42)$$

Factor \tilde{C} into two parts by defining another N by $2N$ matrix \mathcal{C} and a $2N$ by $2N$ diagonal matrix \mathcal{F} by

$$\mathcal{C}_{j,m} = z_m^{j-1} + \alpha_m z_m^{N-j} \quad (3.43)$$

$$\mathcal{F}_{j,m} = -i \delta_{jm} (1 + \alpha_m u z_m^N - u z_m - \alpha_m z_m^{N+1})/z_m^N, \quad (3.44)$$

then $\tilde{C} = \mathcal{C} \mathcal{F}$ and (3.42) becomes

$$Q = \mathcal{C} \mathcal{F} \mathcal{D}^{M-1} \mathcal{W}^{-1} \mathcal{F} \mathcal{C}^T. \quad (3.45)$$

4. The limit of M large

We emphasize that equations (2.35), (3.45) are *exact*, giving the partition function of a finite M by N lattice.

We only need the determinant of Q , but because the matrix C in (3.45) is not square (it is N by $2N$), we cannot see how to simplify $\det Q$ further for finite M .

However, we are interested in the limit of M large (where we can neglect terms of relative order $e^{-\gamma M}$, and M enters (3.45) only via the explicit power of the diagonal matrix \mathcal{D} therein).

Since λ_1 is the largest of the eigenvalues λ_m , when M is large, the elements of Q will be dominated by terms proportional to λ_1^{M-1} . However, this contribution to Q is of rank 1, so has zero determinant. To obtain a non-zero determinant, we must retain at least N distinct eigenvalues from the λ_m so as to ensure that the truncated matrix Q is non-singular. The choice that gives the largest contribution to $\det Q$ is to choose the N largest eigenvalues, i.e. $\lambda_1, \dots, \lambda_N$. For M large and N fixed, the terms that are neglected will be relatively exponentially small, so we should still obtain the first four contributions to $\log Z$, as in (1.1).

Keeping only the first N diagonal elements of \mathcal{D} is the same as truncating $\mathcal{D}, \mathcal{C}, \mathcal{F}, \mathcal{W}$ to the N by N matrices D, C, F, W with elements

$$D_{ij} = \mathcal{D}_{ij}, \quad C_{ij} = \mathcal{C}_{ij}, \quad F_{ij} = \mathcal{F}_{ij}, \quad W_{ij} = \mathcal{W}_{ij} \quad (4.1)$$

for $i, j = 1, \dots, N$.

Set

$$\Delta = \lambda_1 \lambda_2 \cdots \lambda_N, \quad \phi = f_{11} f_{22} \cdots f_{NN}, \quad \hat{\tau} = \tau_1 \tau_2 \cdots \tau_N \quad (4.2)$$

then from (3.45)

$$\det Q = \frac{(\det C)^2 \phi^2 \Delta^{M-1}}{\hat{\tau}}. \quad (4.3)$$

Using (3.36), this becomes

$$\det Q = \frac{\epsilon_N (\det C)^2 \phi^2 \Delta^{M-1}}{\det Y}. \quad (4.4)$$

Let

$$\zeta = z_1 z_2 \cdots z_N, \quad (4.5)$$

then from (3.44)

$$\phi = (-i)^N \zeta^{-N} \prod_{m=1}^N (1 + \alpha_m u z_m^N - u z_m - \alpha_m z_m^{N+1}) \quad (4.6)$$

We define also

$$\phi' = (-i)^N \zeta^{-N} \prod_{m=1}^N (1 - \alpha_m u z_m^N - u z_m + \alpha_m z_m^{N+1}) \quad (4.7)$$

and set

$$\eta = \prod_{m=1}^N \frac{1 + z_m}{1 - z_m}. \quad (4.8)$$

We shall find it convenient to break the RHS of (4.4) into three parts and to write the equation as

$$\det Q = E G \Delta^{M-1}, \quad (4.9)$$

where

$$E = \frac{\epsilon_N \phi \phi' (\det C)^2}{\eta \det Y}, \quad G = \frac{\eta \phi}{\phi'} \quad (4.10)$$

Set

$$z_m = e^{2i\theta_m}, \quad \alpha_m = e^{2ia_m}. \quad (4.11)$$

Then, using (3.8),

$$\log \lambda_m = \cosh^{-1}(c' c^* - s' s^* \cos 2\theta_m) \quad (4.12)$$

and

$$G = \prod_{m=1}^N (\cot \theta_m) \frac{\sin(a_m + N\theta_m + \theta_m) + u \sin(a_m + N\theta_m - \theta_m)}{\cos(a_m + N\theta_m + \theta_m) + u \cos(a_m + N\theta_m - \theta_m)}. \quad (4.13)$$

$\theta_1, \dots, \theta_{N-1}$ are real, between 0 and $\pi/2$, while θ_N is positive pure imaginary; a_m is either zero or $\pi/2$. It follows that $\log \lambda_m$ and G are real. In fact they are positive.

4.1. Calculation of E

We shall show in Appendices A and B that in the M large limit, E is a product of powers of simple rational functions of u and t . The working is messy and we have to consider separately the cases when N is even and when N is odd. The final result is given immediately below in (4.16). The author hopes that someone will find a way to simplify the working.

We first show in Appendix B that the determinants of Y and C as products of simple factors that are linear in z_1, \dots, z_N . From (B11), (B13), (B14),

$$\frac{\det Y}{(\det C)^2} = \frac{\mathcal{L}(1-u^2)^{N-1}}{\zeta^{2N}} \prod_{1 \leq j < m \leq N}^{\dagger\dagger} (z_j - z_m)^2 (1 - z_j z_m)^2, \quad (4.14)$$

where the superfix $\dagger\dagger$ means that the double product is over all j, m of opposite parity, i.e. $j - m = 1$ modulo 2, and

$$\begin{aligned} \mathcal{L} &= \prod_{j=1}^{N/2} (1 + z_{2j-1})^2 (1 - z_{2j})^2 \quad \text{if } N \text{ is even,} \\ &= \frac{1}{2} \epsilon_N \prod_{j=1}^{(N+1)/2} (1 - z_{2j-1}^2)^2 \quad \text{if } N \text{ is odd.} \end{aligned} \quad (4.15)$$

In Appendix C we go on to calculate the E of (4.10). The fact that the double product is restricted to j, m being of opposite parity is significant. Factors with the same parity do occur in $\det Y$ and $\det C$, but cancel out of the ratio (4.14). This means that we only need specific values of the polynomials $P_1(z), P_2(z)$

defined in Appendix C, rather than their derivatives, which are considerably more complicated.

The results for N even and N odd are given in eqns. (C17) and (C22). They are identical: both give

$$E = \frac{(1-t)^{2N}(1-ut)(1-t^2/u^2)^{3/4}}{2^{N-2}(1-t/u)(1-u^2t^2)^{1/4}(1-t^2)^{1/2}} . \quad (4.16)$$

From (3.19) we see that E is positive real. So therefore is the RHS of (4.9).

5. The free energies as integrals

Define

$$\rho(z) = \sqrt{\frac{(1-tuz)(uz-t)}{(z-tu)(u-tz)}} \quad (5.1)$$

where the square root is chosen to be in the right-half plane. Then $\rho(z)$ is analytic in the cut z -plane denoted in Figure 2, where there are branch cuts denoted by solid (red) horizontal lines on the positive real axis, from 0 to t/u , and from u/t to $+\infty$.

Define also functions $\lambda(z), g(z)$ by

$$\lambda(z) + 1/\lambda(z) = 2c'c^* - s's^*(z + 1/z), \quad (5.2)$$

which is eqn. (3.8), and

$$g(z) = \frac{(1+z)[1-uz+(u-z)\rho(z)]}{(1-z)[1-uz-(u-z)\rho(z)]} \quad (5.3)$$

choosing $\lambda(z) > 1$ when $|z| = 1$. They are analytic and non-zero in the cut plane of Figure 2 ($g(z)$ does *not* have zeros or poles at $z = \pm 1$) and have the following symmetry properties:

$$\rho(1/z) = 1/\rho(z), \quad \lambda(1/z) = \lambda(z), \quad g(1/z) = g(z),$$

and

$$\Delta = \prod_{j=1}^N \lambda(z_j), \quad G = \prod_{j=1}^N g(z_j). \quad (5.4)$$

Set

$$h(z) = \log g(z) + (M-1) \log \lambda(z) \quad (5.5)$$

and consider the integral difference

$$\mathcal{I} = \frac{1}{2\pi i} \left\{ \oint_{C_1} \frac{P'(z)h(z)}{P(z)} dz - \oint_{C_2} \frac{P'(z)h(z)}{P(z)} dz \right\} \quad (5.6)$$

where $P(z)$ is the polynomial (3.21), C_1, C_2 are the two circles in Figure 2, shown as dashed (red) lines, z_n is inside C_2 , and z_{n+1} is outside C_1 .

The function $h(z)$ is analytic in the cut z -plane, which includes C_1, C_2 and the region in between. $P(z)$ is the polynomial (3.21), so the only singularities between C_1 and C_2 are the poles at $z = 1, -1, z_1, \dots, z_{N-1}, z_{N+2}, \dots, z_{2N}$.

The corresponding values of $\lambda(z)$ are $\lambda(1), \lambda(-1), \lambda_1, \dots, \lambda_{N-1}, \lambda_{N-1}, \dots, \lambda_1$, and similarly for $g(z)$. (Note that for $j > N, \lambda(z_j) = 1/\lambda_j = \lambda_{2N+1-j}$).

It follows that \mathcal{I} is the sum of the residues at these $2N$ poles, and that

$$\begin{aligned}\mathcal{I} &= h(1) + h(-1) + 2h(z_1) + \dots + 2h(z_{N-1}) \\ &= h(1) + h(-1) - 2h(z_N) + 2\log(G\Delta^{M-1})\end{aligned}\quad (5.7)$$

so

$$\log \det Q = \log E + \frac{1}{2}[\mathcal{I} + 2h(z_N) - h(1) - h(-1)] \quad (5.8)$$

From (3.26), $z_N = t/u$ to within relatively exponentially small terms in N , so

$$\begin{aligned}\lambda(1) &= \frac{(1-t)(1+u)}{(1+t)(1-u)}, \quad \lambda(-1) = \frac{(1+t)(1+u)}{(1-t)(1-u)}, \quad \lambda(z_N) = 1, \\ g(z_N) &= \frac{u+t}{u-t},\end{aligned}$$

$$g(1) = \frac{u(1+u)(1-t)^2}{(1-u)(u-t)(1-tu)}, \quad g(-1) = \frac{(1+u)(u+t)(1+tu)}{u(1-u)(1-t)^2}$$

taking limits $z \rightarrow \pm 1$ in (5.3) as necessary. It follows that

$$2h(z_N) - h(1) - h(-1) = 2M \log \left[\frac{1-u}{1+u} \right] + \log \left[\frac{(u+t)(1-tu)}{(u-t)(1+tu)} \right] \quad (5.9)$$

On C_1 , $|z| > 1$, so for N large, from (3.21), neglecting only exponentially small terms, we can replace $P(z)$ in the integral over C_1 by $z^{2N}(z-ut)(z-u/t)$.

On C_2 , $|z| < 1$, so we can replace $P(z)$ in the integral over C_2 by $(1-tuz)(1-uz/t)$.

We make these substitutions. We can then move C_1, C_2 to each become the unit circle, giving

$$\mathcal{I} = \frac{1}{2\pi i} \oint h(z) \left\{ \frac{2N}{z} + \frac{1}{z-ut} - \frac{t}{u-tz} + \frac{tu}{1-tuz} - \frac{u}{uz-t} \right\} dz \quad (5.10)$$

the integration being round the unit circle. Setting $z = e^{i\theta}$, this becomes the real integral

$$\mathcal{I} = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) \{2N + \sigma(\theta)\} d\theta. \quad (5.11)$$

where

$$\sigma(\theta) = \frac{1-t^2u^2}{1+t^2u^2-2tu\cos\theta} - \frac{u^2-t^2}{t^2+u^2-2tu\cos\theta}. \quad (5.12)$$

From (5.5), $h(z)$ is linear in M , so \mathcal{I} , like the RHS of (1.1), is the sum of four terms, proportional to MN , M , N , and 1, respectively. So are the other terms that enter $\log Z$ via (2.35), (4.16), (5.8). Define

$$\mu(z) = g(z)/\lambda(z), \quad (5.13)$$

so that (5.5) becomes

$$h(z) = M \log \lambda(z) + \log \mu(z). \quad (5.14)$$

Following Onsager[4, eqn. 2.1a], define

$$k = (\sinh 2H \sinh 2H')^{-1} = s^*/s' = \frac{t(1-u^2)}{u(1-t^2)} \quad (5.15)$$

and note that

$$1+k = \frac{(1+t/u)(1-ut)}{u(1-t^2)}, \quad 1-k = \frac{(1-t/u)(1+ut)}{u(1-t^2)}. \quad (5.16)$$

Putting together (2.35), (3.17), (4.9), (5.8), (5.9) and (5.10), and using the above formulae, we obtain

$$\begin{aligned} \log Z &= \frac{MN}{2} \log 2 \sinh 2H - MH' - N(H + H^*) + \log 2 + \\ &\frac{1}{8} \log \left[\frac{(1+k)^5}{(1-k)^3} \right] + \frac{1}{8\pi} \int_0^{2\pi} \left[M \log \lambda(e^{i\theta}) + \log \mu(e^{i\theta}) \right] [2N + \sigma(\theta)] d\theta. \end{aligned} \quad (5.17)$$

This is indeed of the expected form (1.1).

Separating (5.17) into its four constituent terms, we obtain

$$-\beta f_b = \frac{1}{2} \log(2 \sinh 2H) + \frac{1}{4\pi} \int_0^{2\pi} \log \lambda(e^{i\theta}) d\theta \quad (5.18)$$

$$-\beta f_s = -H' + \frac{1}{8\pi} \int_0^{2\pi} \sigma(\theta) \log \lambda(e^{i\theta}) d\theta \quad (5.19)$$

$$-\beta f'_s = -H + H^* + \frac{1}{4\pi} \int_0^{2\pi} \log \mu(e^{i\theta}) d\theta \quad (5.20)$$

$$-\beta f_c = \log 2 + \frac{1}{8} \log \left[\frac{(1+k)^5}{(1-k)^3} \right] + \frac{1}{8\pi} \int_0^{2\pi} \sigma(\theta) \log \mu(e^{i\theta}) d\theta. \quad (5.21)$$

From (5.2),

$$\log \lambda(e^{i\theta}) = \cosh^{-1}(c'c^* - s's^* \cos \theta) \quad (5.22)$$

so (5.18) is Onsager's famous result for the bulk free energy of the Ising model.[4, eqn. 106]

6. The free energies in terms of elliptic functions

Define x, y (not to be confused with the x, y of section 3) by $\lambda(z) = (xy)^{-1}$, $z = x/y$. Then (5.2) becomes

$$x^2 y^2 + 1 + s' s^* (x^2 + y^2) - 2c' c^* xy = 0. \quad (6.1)$$

This is eqn. (15.10.3) of [12]. I show there that such an equation can be parametrised using elliptic functions of modulus k , where

$$k + 1/k = \frac{c'^2 c^{*2} - 1 - s'^2 s^{*2}}{s' s^*} = \frac{s'^2 + s^{*2}}{s' s^*}. \quad (6.2)$$

We choose, using (3.16),

$$k = \frac{s^*}{s'} = \frac{t(1-u^2)}{u(1-t^2)} \quad (6.3)$$

and as usual set $k' = (1-k^2)^{1/2}$.

Let sn , cn , dn be the usual meromorphic elliptic Jacobi functions, and K, K' the complete elliptic integrals, replacing η in [12] by v ,

$$c'c^* = -\frac{\text{cn } v \text{ dn } v}{k \text{ sn}^2 v}. \quad (6.4)$$

From eqn. (15.10.12) of [12], we can introduce a parameter v such that $x = -k^{1/2} \text{sn}(r-v/2)$, $y = -k^{1/2} \text{sn}(r+v/2)$, and hence

$$z = \frac{\text{sn}(r-v/2)}{\text{sn}(r+v/2)}, \quad \lambda(z) = \frac{1}{k \text{sn}(r+v/2) \text{sn}(r-v/2)}. \quad (6.5)$$

In the ordered ferromagnetic regime (3.19),

$$0 < k < 1 \quad \text{and} \quad v, iK' - v = \text{positive pure imaginary}. \quad (6.6)$$

We take r to be real:

$$-K < r \leq K. \quad (6.7)$$

Then λ is real and $\lambda > 1$. As r goes from $-K$ to K , z moves anti-clockwise around the unit circle from 1 through $i, 0, -i$ back to 1.

We can choose, consistently with (6.3), (6.4),

$$c' = \frac{i \text{ dn } v}{k \text{ sn } v}, \quad s' = \frac{i}{k \text{ sn } v}, \quad c^* = \frac{i \text{ cn } v}{\text{sn } v}, \quad s^* = \frac{i}{\text{sn } v}, \quad (6.8)$$

where sn , cn , dn are the usual elliptic Jacobi functions. They satisfy the relations

$$\text{cn}^2(u) = 1 - \text{sn}^2(u), \quad \text{dn}^2(u) = 1 - k^2 \text{sn}^2(u) \quad (6.9)$$

$$\begin{aligned} \frac{d}{du} \text{sn}(u) &= \text{cn}(u) \text{dn}(u), \quad \frac{d}{du} \text{cn}(u) = -\text{sn}(u) \text{dn}(u), \\ \frac{d}{du} \text{dn}(u) &= -k^2 \text{sn}(u) \text{cn}(u). \end{aligned} \quad (6.10)$$

Setting

$$\bar{v} = iK' - v, \quad (6.11)$$

it follows that

$$u = -i \frac{\text{sn}(\bar{v}/2) \text{dn}(\bar{v}/2)}{\text{cn}(\bar{v}/2)}, \quad t = -ik \frac{\text{sn}(\bar{v}/2) \text{cn}(\bar{v}/2)}{\text{dn}(\bar{v}/2)} \quad (6.12)$$

and

$$\frac{1-t}{1+t} = -i \frac{\text{sn}(v/2) \text{dn}(v/2)}{\text{cn}(v/2)}, \quad \frac{1-u}{1+u} = -ik \frac{\text{sn}(v/2) \text{cn}(v/2)}{\text{dn}(v/2)} \quad (6.13)$$

(interchanging H with H' takes v to \bar{v} and u, t to $(1-t)/(1+t), (1-u)/(1+u)$).

One can establish many identities involving z, t, u by using Liouville's theorem, as described in Chapter 15 of [12], and writing the functions sn , cn , dn

as ratios of the entire theta function H, H_1, Θ, Θ_1 . For instance, z is a doubly periodic function of r , of periods $2K, 2iK'$. Because $\text{sn}(iK' - r) = -1/k \text{sn}(r)$, it follows that z is an even function of $r - iK'/2$, and that when $r = iK'/2$, then $z = -k \text{sn}^2(\bar{v}/2) = tu$. Hence $r = iK'/2$ is a double zero of the expression $z - tu$, and a double pole of

$$\rho(z)^2 = \frac{(1 - tuz)(uz - t)}{(z - tu)(u - tz)} \quad (6.14)$$

Proceeding similarly, the poles and zeros of

$$\mathcal{J} = - \left[\frac{\text{cn}(r - iK'/2)}{\text{sn}(r - iK'/2) \text{dn}(r - iK'/2)} \right]^2 \quad (6.15)$$

are also all poles and zeros of $\rho(z)^2$. There are two such double zeros, and two double poles. Because each factor (e.g. $z - tu$) in (6.14) has two zeros in a period rectangle, these are all the poles and zeros of $\rho(z)^2$. The ratio $\rho(z)^2/\mathcal{J}$ is therefore entire and doubly periodic, so it is bounded at infinity and from Liouville's theorem must be a constant. It is one when $r = v/2$ and $z = 0$, so is one for all r . Taking the square root so that $\rho(z)$ is positive real when r is real, we have proved that

$$\rho(z) = -i \frac{\text{cn}(r - iK'/2)}{\text{sn}(r - iK'/2) \text{dn}(r - iK'/2)} \quad (6.16)$$

so is a single-valued meromorphic function of r .

This means that the polynomial $P(z)$ in (3.21) contains the two factors $z^n - \rho(z)$ and $z^n + \rho(z)$, each of which is a single-valued meromorphic function of r . We used a similar factorization property in (C8) of Appendix C to calculate E , but it was only in the limit of N large that we could identify the two factors with the RH sides of (C10), (C11). It's possible that that argument could be made true for finite N if we had used the elliptic function parametrization.

The expression in braces on the RHS of (5.10) is

$$\frac{d}{dz} \log[z^{2N}/\rho(z)^2]$$

so if θ is defined by $z = e^{i\theta}$, then

$$\sigma(\theta) = -2z \frac{d}{dz} \log \rho(z) = -2 \left(\frac{d}{dr} \log \rho \right) \left(\frac{d}{dr} \log z \right)^{-1}. \quad (6.17)$$

Using (6.10) and relations (8.151.2), (8.156) of [22], we can establish that

$$\frac{d}{dr} \log \rho = 2ik \text{cn}(2r) \quad (6.18)$$

and

$$\frac{d}{dr} \log z = - \frac{\text{sn}(v)[1 - k^2 \text{sn}^2(r + v/2) \text{sn}^2(r - v/2)]}{\text{sn}(r + v/2) \text{sn}(r - v/2)} \quad (6.19)$$

Also,

$$\frac{1 - uz + (u - z)\rho(z)}{1 - uz - (u - z)\rho(z)} = \frac{i \lambda(z) \text{cn}(r)}{\text{sn}(r) \text{dn}(r)} \quad (6.20)$$

$$\frac{1+z}{1-z} = \frac{\operatorname{cn}(v/2) \operatorname{dn}(v/2) \operatorname{sn}(r)}{\operatorname{sn}(v/2) \operatorname{cn}(r) \operatorname{dn}(r)}, \quad (6.21)$$

so from (5.3), (5.13),

$$\mu(z) = \frac{i \operatorname{cn}(v/2) \operatorname{dn}(v/2)}{\operatorname{sn}(v/2) \operatorname{dn}^2(r)}. \quad (6.22)$$

Regard λ, z, ρ as functions of the elliptic argument r and define

$$\begin{aligned} A_1(r) &= \log \lambda, \quad A_2(r) = \log k' - 2 \log \operatorname{dn}(r), \\ B_1(r) &= -i \frac{d}{dr} \log z, \quad B_2(r) = i \frac{d}{dr} \log \rho. \end{aligned} \quad (6.23)$$

Set

$$\xi = \log(1-t^2) - \frac{1}{2} \log k' + \frac{1}{2} \log \frac{i \operatorname{cn}(v/2) \operatorname{dn}(v/2)}{\operatorname{sn}(v/2)}, \quad (6.24)$$

then (5.18) - (5.21) become

$$\begin{aligned} -\beta f_b &= \frac{1}{2} \log(2 \sinh 2H) + \frac{1}{4\pi} \int_{-K}^K A_1(r) B_1(r) dr, \\ -\beta f_s &= -H' + \frac{1}{4\pi} \int_{-K}^K A_1(r) B_2(r) dr, \\ -\beta f'_s &= -H + \log \left[\frac{\operatorname{dn}(v/2)}{\sqrt{k'}} \right] + \frac{1}{4\pi} \int_{-K}^K A_2(r) B_1(r) dr, \\ -\beta f_c &= \log 2 + \frac{1}{8} \log \frac{(1+k)^5}{(1-k)^3} + \frac{1}{4\pi} \int_{-K}^K A_2(r) B_2(r) dr. \end{aligned} \quad (6.25)$$

6.1. The integrals as elliptic-type sums

To evaluate the integrals in (5.18) - (5.21), we expand $\log z, \log \rho, \log \lambda, \log \mu$, as sums, using the product expansions (15.1.5), (15.1.6) of [12] and taking logarithms. Setting

$$q = e^{-\pi K'/K}, \quad w = e^{i\pi v/2K} \quad (6.26)$$

we get

$$\begin{aligned} A_1(r) &= \frac{\pi(K' + iv)}{2K} + 2 \sum_{m=1}^{\infty} \frac{(w^m - q^m w^{-m}) \cos(\pi m r / K)}{m(1 + q^m)}, \\ A_2(r) &= -8 \sum_{m \text{ odd}}^{\infty} \frac{q^m \cos(\pi m r / K)}{m(1 - q^{2m})}, \\ B_1(r) &= \frac{\pi}{K} + \frac{2\pi}{K} \sum_{m=1}^{\infty} \frac{(w^m + q^m w^{-m}) \cos(\pi m r / K)}{1 + q^m}, \\ B_2(r) &= \frac{4\pi}{K} \sum_{m \text{ odd}}^{\infty} \frac{q^{m/2} \cos(\pi m r / K)}{1 + q^m}, \end{aligned} \quad (6.27)$$

where the subscript “m odd” in the sums in the second and fourth equations means that the sums are over all odd positive integers 1, 3, 5, etc. Similarly throughout this paper. All series in this and the next section are convergent in an annulus not smaller than $q^{1/2} < |w| < 1$.

Substituting these Fourier series into (6.25), we obtain

$$\begin{aligned}
-\beta f_b &= \frac{1}{2} \log(2 \sinh 2H) + \frac{\pi(K' + iv)}{4K} + \sum_{m=1}^{\infty} \frac{w^{2m} - q^{2m} w^{-2m}}{m(1 + q^m)^2} \\
-\beta f_s &= -H' + 2 \sum_{m \text{ odd}}^{\infty} \frac{q^{m/2}(w^m - q^m w^{-m})}{m(1 + q^m)^2} \\
-\beta f'_s &= -H + \log \frac{\text{dn}(v/2)}{\sqrt{k'}} - 4 \sum_{m \text{ odd}}^{\infty} \frac{q^m(w^m + q^m w^{-m})}{m(1 + q^m)^2(1 - q^m)} \\
-\beta f_c &= \log 2 + \frac{1}{8} \log \frac{(1+k)^5}{(1-k)^3} - 8 \sum_{m \text{ odd}}^{\infty} \frac{q^{3m/2}}{m(1 + q^m)^2(1 - q^m)}.
\end{aligned} \tag{6.28}$$

Using (15.1.5) and (15.1.6) of [12],

$$\log \frac{\text{dn}(v/2)}{\sqrt{k'}} = 2 \sum_{m \text{ odd}}^{\infty} \frac{q^m(w^m + w^{-m})}{m(1 - q^{2m})}. \tag{6.29}$$

From (3.15)

$$H^* = -\frac{1}{2} \log \tanh H = -\frac{1}{2} \log \left(\frac{1-t}{1+t} \right) \tag{6.30}$$

so by using (6.13),

$$\begin{aligned}
H^* &= \sum_{m \text{ odd}}^{\infty} \frac{w^m - q^m w^{-m}}{m(1 + q^m)}, \\
H &= -\frac{i\pi v}{4K} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{q^m(w^{-2m} - w^{2m})}{m(1 + q^{2m})}, \\
H' &= \frac{\pi(K' + iv)}{4K} + \frac{1}{2} \sum_{m=1}^{\infty} \frac{w^{2m} - q^{2m} w^{-2m}}{m(1 + q^{2m})}
\end{aligned} \tag{6.31}$$

and

$$\log 2 \sinh(2H) = \frac{1}{2} \log \left(\frac{4}{k} \right) - \frac{\pi(K' + 2iv)}{4K} - \sum_{m=1}^{\infty} \frac{w^{2m} - q^m w^{-2m}}{m(1 + q^m)} \tag{6.32}$$

Also, from (15.1.4a) and (15.6.5) of [12],

$$k' = (1 - k^2)^{1/2} = \prod_{m=1}^{\infty} \left(\frac{1 - q^{2m-1}}{1 + q^{2m-1}} \right)^4 \tag{6.33}$$

$$\frac{1-k}{1+k} = \hat{k}' = \prod_{m=1}^{\infty} \left(\frac{1 - q^{m-1/2}}{1 + q^{m-1/2}} \right)^4, \tag{6.34}$$

where if we regard k' as a function $k'(q)$ of q , then $\hat{k}' = k'(q^{1/2})$. Using these formulae, we can simplify (6.28) to

$$\begin{aligned}
-\beta f_b &= \frac{1}{2} \log(2 \sinh 2H) + \frac{\pi(K' + iv)}{4K} + \sum_{m=1}^{\infty} \frac{w^{2m} - q^{2m} w^{-2m}}{m(1 + q^m)^2} \\
-\beta f_s &= -H' + 2 \sum_{m \text{ odd}}^{\infty} \frac{q^{m/2}(w^m - q^m w^{-m})}{m(1 + q^m)^2} \\
-\beta f'_s &= -H - H^* + \sum_{m \text{ odd}}^{\infty} \frac{(1 - q^m)(w^m + q^m w^{-m})}{m(1 + q^m)^2} \\
-\beta f_c &= \log 2 + \frac{1}{8} \log \frac{(1 + k)^3}{1 - k} + 2 \sum_{m \text{ odd}}^{\infty} \frac{q^{m/2}(1 - q^m)}{m(1 + q^m)^2}.
\end{aligned} \tag{6.35}$$

6.2. Final results

Finally, we can simplify the formulae to

$$\begin{aligned}
-\beta f_b &= H + H' + \sum_{m=1}^{\infty} \frac{q^m(1 - q^m)(w^m - q^m w^{-m})(w^{-m} - w^m)}{m(1 + q^m)^2(1 + q^{2m})} \\
-\beta f_s &= -H' + 2 \sum_{m \text{ odd}}^{\infty} \frac{q^{m/2}(w^m - q^m w^{-m})}{m(1 + q^m)^2} \\
-\beta f'_s &= -H - 2 \sum_{m \text{ odd}}^{\infty} \frac{q^m(w^m - w^{-m})}{m(1 + q^m)^2} \\
-\beta f_c &= \log 2 + \frac{1}{4} \log k' + 4 \sum_{m \text{ odd}}^{\infty} \frac{q^{m/2}(1 + q^{2m})}{m(1 + q^m)^2(1 - q^m)}.
\end{aligned} \tag{6.36}$$

This is quite a natural way of writing the results, the preliminary terms linear in H, H' being the zero-temperature terms in a series expansion, starting from the state(s) with all spins equal.

For the surface and corner free energies, this is also mathematically a natural way to write the results, as the sums are anti-symmetric either in negating $q^{1/2}$ while keeping w fixed, or in negating w while keeping q fixed; e^{4H} , $e^{4H'}$ and k' are unchanged by such negations.

This implies some quite mysterious properties of the surface and corner free energies. Remembering that the spontaneous magnetization of the Ising model is [23, 24, 25]

$$\mathcal{M}_0 = k'^{1/4} = (1 - k^2)^{1/8},$$

then considered as functions of $p = q^{1/2}$ and w they satisfy

$$\begin{aligned}
f_s(-p, w) &= f_s(p, -w) \\
-\beta f_s(p, w) - \beta f_s(-p, w) &= \frac{i\pi}{2} - 2H' \\
-\beta f'_s(p, w) - \beta f'_s(-p, w) &= \frac{i\pi}{2} - 2H \\
\exp[-\beta f_c(p) - \beta f_c(-p)] &= 4\mathcal{M}_0^2.
\end{aligned}$$

6.3. Product forms

Taking exponentials, the right-hand sides of (6.36) become products. In particular, setting

$$p = q^{1/2}, \quad s_n = \left(\frac{(1 - q^n w^2)(1 - q^{n+1} w^{-2})}{(1 - q^n)(1 - q^{n+1})} \right)^n,$$

we find that

$$\begin{aligned} e^{-\beta f_b} &= e^{H+H'} \prod_{n=1}^{\infty} \frac{s_{2n-1} (1 - q^{4n-2})(1 - q^{4n-1} w^2)(1 - q^{4n} w^{-2})}{s_{2n} (1 - q^{4n})(1 - q^{4n-2} w^2)(1 - q^{4n-1} w^{-2})} \\ e^{-\beta f_s} &= e^{-H'} \prod_{n=1}^{\infty} \left(\frac{(1 + p^{4n-3} w)(1 - p^{4n-1} w^{-1})}{(1 - p^{4n-3} w)(1 + p^{4n-1} w^{-1})} \right)^{2n-1} \left(\frac{(1 - p^{4n-1} w)(1 + p^{4n+1} w^{-1})}{(1 + p^{4n-1} w)(1 - p^{4n+1} w^{-1})} \right)^{2n} \\ e^{-\beta f'_s} &= e^{-H} \prod_{n=1}^{\infty} \left(\frac{(1 + q^{2n-1} w^{-1})(1 - q^{2n-1} w)}{(1 - q^{2n-1} w^{-1})(1 + q^{2n-1} w)} \right)^{2n-1} \left(\frac{(1 - q^{2n} w^{-1})(1 + q^{2n} w)}{(1 + q^{2n} w^{-1})(1 - q^{2n} w)} \right)^{2n} \\ e^{-\beta f_c} &= 2 \frac{(1+k)^{3/8}}{(1-k)^{1/8}} \prod_{n=1}^{\infty} \left(\frac{1 + p^{4n-3}}{1 - p^{4n-3}} \right)^{4n-3} \left(\frac{1 - p^{4n-1}}{1 + p^{4n-1}} \right)^{4n-1}. \end{aligned} \quad (6.37)$$

In the isotropic case, when

$$v = iK'/2, \quad w = q^{1/4}, \quad (6.38)$$

then by using (6.33) and (6.34) we can verify that these equations agree with Vernier and Jacobsen's conjectures [1, eqn. 49], except that their q is our p and in the last equation we have included the factor of 2 that comes from the fact that for every contribution in (2.1) to Z from a particular configuration σ of the spins on the lattice, there is another equal contribution from the spins $-\sigma$.

7. The inversion and rotation relations

We emphasize that our results (6.36) have been obtained by arguments that are rigorous, or at least could be made so. Here we present some plausible, but not rigorous, arguments that could have been used to obtain f_b, f_s, f'_s much more easily. They also tell us that f_c does not depend on the anisotropy parameter v (or w).

Set

$$T = V_1^{1/2} V_2 V_1^{1/2}, \quad (7.1)$$

taking $V_1^{1/2}$ to be the positive real square root of V_1 . Then the partition function is

$$Z = \xi^T V_2 V_1 V_2 \cdots V_1 V_2 \xi = \xi^T V_1^{-1/2} T^M V_1^{-1/2} \xi, \quad (7.2)$$

where ξ is the vector with all entries +1. Since

$$V_1 \xi = (2 \cosh H)^N \xi,$$

it follows that

$$Z = (2 \cosh H)^{-N} \xi^T T^M \xi. \quad (7.3)$$

In the physical regime, when v is pure imaginary, between 0 and iK' , the matrix T is real and symmetric. Hence when M is large, neglecting only terms that are relatively exponentially small in M ,

$$Z = (2 \cosh H)^{-N} \Lambda^M \langle 0 | \xi \rangle^2, \quad (7.4)$$

where Λ is the maximum eigenvalue of T and $|0\rangle$ is the corresponding eigenvector. M only enters this equation explicitly, so from (1.1), it follows that

$$e^{-N\beta f_b - \beta f_s} = \Lambda, \quad e^{-N\beta f'_s - \beta f_c} = (2 \cosh H)^{-N} \langle 0 | \xi \rangle^2. \quad (7.5)$$

It has long been known that the bulk free energy of the solvable models can usually be obtained quite simply by the “inversion relation method” ([9], sections 13.6, 14.3, 14.4 of [12]. From (3.16), (6.8), if we regard H, H', V_1, V_2, T as functions of v , then

$$H(2iK' - v) = H(v) + i\pi/2, \quad H'(2iK' - v) = -H'(v)$$

$$V_1(v) V_1(2iK' - v) = (2i \sinh 2H)^N \mathbf{1}, \quad V_2(v) V_2(2iK' - v) = \mathbf{1}, \quad (7.6)$$

and hence

$$T(v) T(2iK' - v) = (2i \sinh 2H)^N \mathbf{1}. \quad (7.7)$$

At $v = iK$, $H = \infty$ and $H' = 0$, so both V_1, V_2, T are all proportional to the identity matrix. Hence as v moves through the point iK , the eigenvalues all become equal and cross over one another, the largest becoming the smallest. If v is below the inversion point iK' , then $\Lambda(v)$ is the largest eigenvalue and $\Lambda(2iK' - v)$ is the smallest. It is reasonable to suppose that $\Lambda(2iK' - v)$ is the analytic continuation of $\Lambda(v)$, and from (7.7) that

$$\Lambda(v) \Lambda(2iK' - v) = (2i \sinh 2H)^N. \quad (7.8)$$

From (7.7), $T(2iK' - v)$ commutes with $T(v)$, so has the same eigenvectors and $|0\rangle$ is unchanged.

This all fits with our matrix representatives calculation of section 3. If $v \rightarrow 2iK' - v$, then $\widehat{V}_1, \widehat{V}_2$ are inverted and $\bar{v} \rightarrow -\bar{v}$. From (6.12) $u, t \rightarrow -u, -t$ and from (3.21) we can leave z_1, \dots, z_N unchanged. If we also leave $\alpha_1, \dots, \alpha_N$ unchanged, then $\lambda_1, \dots, \lambda_N$ are inverted and we can verify that for each eigenvector \mathbf{x} is interchanged with \mathbf{y} , which leaves (3.2) unchanged. This is equivalent to leaving the eigenvectors of $\widehat{V}_1^{1/2} \widehat{V}_2 \widehat{V}_1^{1/2}$ unchanged, but inverting their eigenvalues.

Considering the first of the relations (7.5) when v has its original value and when it is replaced by $2iK' - v$, and taking the product, we obtain relations for

f_b and f_s . Similarly, considering the second of the relations and taking ratios, we obtain relations for f'_s and f_c . They are

$$\begin{aligned} -\beta f_b(v) - \beta f_b(2iK' - v) &= i\pi/2 + \log[2 \sinh 2H] \\ -\beta f_s(v) - \beta f_s(2iK' - v) &= 0 \\ -\beta f'_s(v) + \beta f'_s(2iK' - v) &= i\pi/2 + \log \tanh H \\ -\beta f_c(v) + \beta f_c(2iK' - v) &= 0. \end{aligned} \quad (7.9)$$

Replacing v by $iK' - v$ is equivalent to interchanging H with H' and hence to rotating the lattice through 90° , which gives the following four relations

$$\begin{aligned} f_b(iK' - v) &= f_b(v) \quad , \quad f_s(iK' - v) = f'_s(v) \, , \\ f'_s(iK' - v) &= f_s(v) \quad , \quad f_c(iK' - v) = f_c(v) \, . \end{aligned} \quad (7.10)$$

We can verify that our results (6.36) do indeed satisfy these relations.

Series expansions also suggest that $-\beta f_b - H - H'$, $-\beta f_s + H'$, $-\beta f'_s + H$, $-\beta f_c$ are analytic functions of v , not just in the physical regime $0 < \text{Im } v < K'$, but in the extended regime $-\epsilon < \text{Im } v < K' + \epsilon$, where ϵ is positive (but less than K'). They also suggest that the four functions are periodic in v of period at most $4K$, so they are Laurent expandable in powers of w .

These observations almost define the four free energies, as we shall now show. The last imply that there exist expansions of the form

$$\begin{aligned} -\beta f_b &= H + H' + \sum_{m=-\infty}^{\infty} c_{1,m} w^m \quad , \quad -\beta f_s = -H' + \sum_{m=-\infty}^{\infty} c_{2,m} w^m \\ -\beta f'_s &= -H + \sum_{m=-\infty}^{\infty} c_{3,m} w^m \quad , \quad -\beta f_c = \sum_{m=-\infty}^{\infty} c_{4,m} w^m \end{aligned} \quad (7.11)$$

which are convergent for $q^{1/2} < |w| < 1$.

Substituting these into the equations (7.10) and (7.9) for f_b , using the identities

$$\log[2 \sinh 2H] - 2H = \sum_{m=1}^{\infty} \frac{(1 - q^m)(2q^m - w^{2m} - q^{2m}w^{-2m})}{m(1 + q^m)(1 + q^{2m})} \quad (7.12)$$

$$\log \tanh H = -2 \sum_{m \text{ odd}}^{\infty} \frac{w^m - q^m w^{-m}}{m(1 + q^m)} \, , \quad (7.13)$$

and equating coefficients in the Laurent expansions, we obtain

$$c_{1,m} = q^{-m/2} c_{1,-m} \quad (7.14)$$

and, for $m \neq 0$,

$$c_{1,m} + q^{-m} c_{1,-m} = \frac{-2(1 - q^{m/2})}{m(1 + q^{m/2})(1 + q^m)} \quad \text{if } m \text{ is even} \, ,$$

while $c_{1,m} + q^{-m} c_{1,-m} = 0$ if m is odd. The case $m = 0$ gives

$$2 c_{1,0} = 2 \sum_{n=1}^{\infty} \frac{q^n(1 - q^n)}{n(1 + q^n)(1 + q^{2n})}$$

Solving these equations, we find that $c_{1,0} = 0$ if m is odd, while if m is even and $m \neq 0$,

$$c_{1,m} = \frac{-2q^{m/2}(1 - q^{m/2})}{m(1 + q^{m/2})^2(1 + q^m)}. \quad (7.15)$$

Substituting these results (for m positive, zero and negative) back into (7.11), we obtain the result (6.36).

Similarly, using (7.10) and (7.9) for f_s, f'_s , we get the equations

$$\begin{aligned} c_{3,m} &= q^{-m/2} c_{2,-m}, \quad c_{2,m} + q^{-m} c_{2,-m} = 0, \\ c_{3,m} - q^{-m} c_{3,-m} &= \frac{-2}{m(1 + q^m)} \quad \text{if } m \text{ is odd,} \quad = 0 \quad \text{if } m \text{ is even.} \end{aligned}$$

Solving these gives

$$c_{2,m} = \frac{2q^{m/2}}{m(1 + q^m)^2}, \quad c_{3,m} = \frac{-2q^m}{m(1 + q^m)^2} \quad \text{if } m \text{ is odd,} \quad (7.16)$$

while $c_{2,m} = c_{3,m} = 0$ if m is even. This also agrees with (6.36).

Finally, for f_c , (7.10) and (7.9) give

$$c_{4,m} = q^{-m/2} c_{4,-m} = q^{-m} c_{4,-m}. \quad (7.17)$$

These equations imply that

$$c_{4,m} = 0 \quad \text{if } m \neq 0, \quad (7.18)$$

so f_c is indeed independent of v and w , as we found. These arguments do *not* give the value of $c_{4,0}$, i.e. the constant term in the Laurent expansion (7.11) of f_c .

Our derivation in this section is *not* rigorous, because we have assumed the existence of the Laurent expansions (7.11). For the Ising model on the square lattice rotated through 45° , with cylindrical boundary conditions, the row-to-row transfer matrices *commute* (because of the Yang-Baxter relations). This means that their eigenvalues, like the Boltzmann weights $e^{-2H}, e^{-2H'}$, are meromorphic functions of v , even for a finite number N of columns. One can then establish rigorously an inversion identity, and from that calculate f_b .

However, we have used closed boundary conditions and the orientation of Figure 1, so our transfer matrices do *not* commute. (The eigenvectors depend on z_1, \dots, z_N , the zeros of the polynomial $P(z)$ in (3.21), which certainly depends on v .) We do not have any *a priori* reason for believing the surface and corner free energies to be meromorphic functions of v .

Indeed, for unsolved models, such as the Ising model in a magnetic field, one can establish inversion and rotation relations like (7.9) and (7.10), but the free energies have complicated singularities at the inversion points and one does not have useful expansions like (7.11).

Having said this, O'Brien, Pearce, Behrend and Batchelor [14, 15, 16] have obtained surface free energies for various solved models by using the ‘‘reflection Yang-Baxter relations’’. These lead to commutation properties of transfer matrices, and Pearce has used the resulting inversion identities to obtain the surface free energies of the self-dual Potts model. The relation 52 of his notes [17] is so like our relation (7.9) for f_s, f'_s that it must be possible to obtain a more rigorous derivation by his methods.

8. The corner free energy

The non-rigorous, but comparatively simple, arguments of the previous section do not give us any information on the corner free energy f_c , beyond telling us that it does not depend on the anisotropy parameter v .

However, there is one intriguing point that gives some hope that it may be possible to obtain it, at least to within simple additive algebraic functions of k .

The integrands in the four equations of (6.25) are $A_1B_1, A_1B_2, A_2B_1, A_2B_2$, respectively. This directly leads to the fact that

$$S_1S_4 = S_2S_3, \quad (8.1)$$

where S_1, S_2, S_3, S_4 are the summands (including the external numerical factors) in the series in the four equations (6.28).

The same is true of the summands in (6.35) and (6.36). (The second form of the equations be obtained, at least formally, by re-defining the function $A_2(r)$, while the third needs a re-definition of both $A_2(r)$ and $B_1(r)$.) Thus if one can obtain the bulk and surface free energies, this will give S_1, S_2, S_3 , so S_4 can then be obtained from (8.1). Considered as functions of q , these summands all have a double pole at $q^m = -1$, and it is this double pole that makes the free energies non-algebraic functions of the Boltzmann weights. It ensures that they are products of factors such as $1 - q^{2n}w$, with exponents that are *linear* in n (rather than constants), as in (6.37).

9. Critical behaviour

9.1. Bulk free energy

The Ising model is critical when $k = 1$ and $q = 1$. We can obtain the behaviour near criticality by using the Poisson transform given in (15.8) of [12]:

$$\delta \sum_{n=-\infty}^{\infty} f(n\delta) = \sum_{n=-\infty}^{\infty} g(2\pi n/\delta), \quad (9.1)$$

where

$$g(k) = \int_{-\infty}^{\infty} e^{ikx} f(x) dx, \quad (9.2)$$

true for any function $f(x)$ that is analytic on the real axis and for which the integral is absolutely convergent.

If we take

$$f(x) = \frac{\pi \sinh 2\alpha x / \pi}{2\alpha x (\cosh x)^2}, \quad (9.3)$$

then $f(x), g(k)$ are even functions, so we can write (9.1) as

$$\delta[f(0) + 2 \sum_{n=1}^{\infty} f(n\delta)] = g(0) + 2 \sum_{n=1}^{\infty} g(2\pi n), \quad (9.4)$$

For $k > 0$ we can close the integration in (9.2) round the upper half x -plane and sum over the residues of the poles at $x = i(2m-1)\pi/2$. This gives $g(k)$ as

a sum over m , the summand being a sum of terms that are either exponential in k , or proportional to the same exponential multiplied by k .

It follows that we can perform the summation on the RHS of (9.4) for each pole, giving

$$\delta + \sum_{n=1}^{\infty} \frac{\pi \sinh(2\alpha\delta n/\pi)}{\alpha n \cosh^2(\delta n)} = g(0) + R_1 + R_2, \quad (9.5)$$

where (replacing $2m-1$ by n)

$$\begin{aligned} R_1 &= 8 \sum_{n \text{ odd}}^{\infty} \frac{y^{2n}}{n(1-y^{2n})} \left[\frac{\sin(n\alpha)}{n\alpha} - \cos(n\alpha) \right], \\ R_2 &= \frac{8\pi^2}{\alpha\delta} \sum_{n \text{ odd}}^{\infty} \frac{y^{2n} \sin(n\alpha)}{n(1-y^{2n})^2}, \end{aligned} \quad (9.6)$$

and

$$y = e^{-\pi^2/2\delta}.$$

Setting

$$\delta = \frac{\pi K'}{2K}, \quad \alpha = \frac{\pi(K' + iv)}{K'}, \quad (9.7)$$

we see that α is real, $\pi/2 < \alpha < \pi$, and the first equation of (6.28) can be written

$$-\beta f_b = \frac{1}{2} \log(2 \sinh 2H) + \frac{\alpha I}{2\pi}, \quad (9.8)$$

where I is the LHS of (9.5) and

$$y = e^{-\pi K/K'} = q'.$$

The function $k'(q')$ is the same as $k(q)$, so from (15.1.4a) of [12],

$$k' = 4q'^{1/2} \prod_{n=1}^{\infty} \left(\frac{1 + q'^{2n}}{1 + q'^{2n-1}} \right)^4.$$

Let $\Delta T = T_c - T$, where T is the temperature and T_c its value at criticality. Then near criticality $1 - k$ vanishes and is proportional to ΔT . Hence so are $k'^2 = 1 - k^2$ and q' . In fact q' is an analytic function of T , with a simple zero at T_c .

The parameter α is non-zero and analytic at T_c , while $\delta = -\pi^2/(2 \log q')$ diverges logarithmically. Substituting the RHS of (9.5) for I in (9.8), we find that the only term that is not analytic is the last, i.e. $\alpha R_2/2\pi$, and the dominant singular contribution to $-\beta f_b$ is

$$\frac{4\pi q'^2 \sin \alpha}{\delta} = -\frac{8q'^2 \log q' \sin \alpha}{\pi}.$$

The internal energy is proportional to the first derivative of f_b with respect to T , and the specific heat to the second derivative. We see that the bulk free energy and internal energy are finite when $T = T_c$, but the specific heat diverges logarithmically, as found by Onsager [4], [12, eqn (7.12.10)].

9.2. Surface and corner free energies

We can restrict the sum in (9.1) to even n by replacing δ by 2δ and dividing by 2. If we then subtract the result from the original equation we obtain another identity:

$$\sum_{n \text{ odd}}^{\infty} \frac{\sinh(2\alpha\delta n/\pi)}{n \cosh^2(\delta n)} = \alpha \left[\frac{1}{2}g(0) + R'_1 + R'_2 \right] / \pi, \quad (9.9)$$

where

$$\begin{aligned} R'_1 &= -4 \sum_{n \text{ odd}}^{\infty} \frac{y^n}{n(1+y^n)} \left[\frac{\sin(n\alpha)}{n\alpha} - \cos(n\alpha) \right], \\ R'_2 &= -\frac{2\pi^2}{\alpha\delta} \sum_{n \text{ odd}}^{\infty} \frac{y^n \sin(n\alpha)}{n(1+y^n)^2}, \end{aligned} \quad (9.10)$$

and $g(x), y$ are defined as before. In these equations, as throughout this paper, the subscript “ n odd” means that the sum is over all odd *positive* integer values of n , i.e. $n = 1, 3, 5$, etc.

If we now define α , not by (9.7), but by

$$\alpha = \frac{\pi(K' + iv)}{2K'}, \quad (9.11)$$

then the last term in the second equation of (6.28) is the LHS of (9.9), so

$$-\beta f_s = -H' + \alpha \left[\frac{1}{2}g(0) + R'_1 + R'_2 \right] / \pi. \quad (9.12)$$

As for the bulk free energy, all the terms on the RHS of this equation are analytic functions of q' at $q' = 0$, except for the δ in R'_2 , so the dominant singularity in $-\beta f_s$ is

$$\frac{q' \log q' \sin \alpha}{\pi} \quad (9.13)$$

and we see that the first derivative of f_s (the “surface internal energy”) diverges logarithmically. Since f_s, f'_s differ only in replacing v by $\bar{v} = iK' - v$ and α by $\pi/2 - \alpha$, the same is true of f' .

The sum in the last equation of (6.35) is the same as that in the second, but with $w = 1$, i.e. $v = 0, \alpha = \pi/2$, so that has the same logarithmic singularity as (9.13). However, the second term the equation contains a contribution

$$-\frac{\log(1-k)}{8}$$

Since $1 - k = 4q'$ when q' is small, the corner free energy itself diverges logarithmically near criticality.

If we define a critical exponent $\hat{\alpha}$ in the usual way [12, eqn 1.7.10b] so that the free energy near $T = T_c$ has a singularity proportional to $(T_c - T)^{2-\hat{\alpha}}$, or in this case (where $\hat{\alpha}$ is an integer)

$$(T_c - T)^{2-\hat{\alpha}} \log(T_c - T)$$

then $\hat{\alpha}$ has the values 0, 1, 2 for the bulk, surface and corner free energy, respectively.

10. Summary

Prompted by the conjectures of Vernier and Jacobsen[1] for the isotropic case, we have used the spinor method of Kaufman[11] to calculate the bulk, surface and corner free energies of the two-dimensional anisotropic Ising model. We do indeed find agreement with [1] for the isotropic case.

The bulk free energy was calculated by Onsager in 1944[4], and the surface free energy by McCoy and Wu in 1967 [7, eqn.4.24b][8, p.126, eqn.4.24b] We have used Kaufman's method to calculate these, together with the corner free energy f_c . This last is by far the most difficult to obtain, involving (unlike f_b and f_s), the calculation of E , G and $A_2(r)$, to which much of this paper, including the whole of the two appendices, is devoted.

We emphasize that our full derivation is rigorous (or at least could be made so), except in section 7. The purpose of that section is to show how one can obtain f_b, f_s, f'_s much more easily if one is prepared to make assumptions about their analyticity properties, and that one also can show that f_c is independent of the anisotropy parameter v (or w). In that respect f_c (for the rectangular lattice) is similar to the order parameter (i.e. the spontaneous magnetization) M_0 .

An intriguing point to which we have referred is that there is structure in the four equations (6.25) for f_b, f_s, f'_s, f_c , coming from the fact that the integrand in (5.17) is a product of two terms, one linear in M , the other in N . This means that the free energy summands S_1, S_2, S_3, S_4 in each of (6.28), (6.35) and (6.36) satisfy the relation (8.1), i.e. $S_1 S_4 = S_2 S_3$, so the summand for the corner free energy is obtainable, using this relation, from those for the bulk and surface free energies. If this could be justified, and the additional simple algebraic terms in, say, equation (6.36), explained, then we could eliminate the lengthy calculation herein of the corner free energy.

11. Acknowledgements

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A. The formula of McCoy and Wu

The relevant low-temperature result of McCoy and Wu is given in eqn. (4.24b) of [7], and in eqn. (4.24b) of page 126 of [8]. Taking $\beta E_1, \beta E_2$ therein to be

our H, H' , defining

$$z_1 = \tanh H, \quad z_2 = \tanh H', \quad \alpha_1 = z_1 e^{-2H'}, \quad \alpha_2 = e^{-2H'}/z_1, \quad (\text{A1})$$

and making some minor adjustments of notation, it is

$$-\beta f_s = -\log[2 \cosh H'] - \frac{i}{\pi} \int_{-i\infty}^{+i\infty} \frac{d\omega}{1-\omega^2} \log[1 - U^{1/2}], \quad (\text{A2})$$

where

$$U = \frac{(\tau_1 - \omega)(\tau_2 - \omega)}{(\tau_1 + \omega)(\tau_2 + \omega)}, \quad \tau_1 = \frac{1 - \alpha_1}{1 + \alpha_1}, \quad \tau_2 = \frac{1 - \alpha_2}{1 + \alpha_2}. \quad (\text{A3})$$

McCoy and Wu have a factor $1/2$ inside the second logarithm in (A2): we have shifted its contribution into the first logarithm, where it gives the factor 2. The square root is to be chosen so that $U^{1/2}$ is one when $\omega = 0$, and continuous on and near the imaginary axis. The symbols $\omega, z_1, z_2, \alpha_1, \alpha_2, \tau_1, \tau_2, U$ used in this Appendix are *not* to be confused with any similar symbols elsewhere in this paper.

We fix the definitions of the integrands in (A2) and (A6) below by first shifting the contour of integration to just to the right of the imaginary axis (thereby avoiding the logarithmic singularities at $\omega = 0$ and ∞), requiring $\log(1 - U^{1/2})$ and $\log(1 + U^{1/2})$ to be real when ω is real and $0 < \omega < \tau_1$, and to be continuous in the right-half ω -plane, except only across the cut on the real axis from τ_1 to τ_2 . This ensures that the integrand when ω is conjugated is the conjugate of the original integrand, and the integrals (after including the multiplication by i) are real.

We rewrite $\log[1 - U^{1/2}]$ as

$$\frac{1}{2} \log[1 - U] + \frac{1}{2} \log \frac{1 - U^{1/2}}{1 + U^{1/2}} \quad (\text{A4})$$

and note that

$$1 - U = \frac{2(\tau_1 + \tau_2)\omega}{(\tau_1 + \omega)(\tau_2 + \omega)}.$$

Consider the contribution of the first term in (A4) to the RHS of (A2). The function $\log(1 - U)$ is analytic in the RH ω -plane. We can close the contour of integration round the RH plane and the only singularity is a simple pole at $\omega = 1$, with residue

$$-\frac{i}{4\pi} \log(1 - \alpha_1 \alpha_2) = -\frac{i}{4\pi} \log(1 - e^{-4H'})$$

Noting that the integration is clockwise round the contour, we obtain a contribution to (A2) of $\frac{1}{2} \log(1 - e^{-4H'})$ and (A2) becomes

$$-\beta f_s = -H' + \frac{1}{2} \log \tanh H' - \mathcal{J}, \quad (\text{A5})$$

where

$$\mathcal{J} = \frac{i}{2\pi} \int_{-i\infty}^{+i\infty} \frac{d\omega}{1-\omega^2} \log \left[\frac{1 - U^{1/2}}{1 + U^{1/2}} \right], \quad (\text{A6})$$

If we define χ so that

$$\omega = \tau_1 \frac{\chi - 1}{\chi + 1}, \quad (\text{A7})$$

then as ω goes from $-i\infty$ to $+i\infty$ along the imaginary axis, χ goes anticlockwise once round the unit circle, from -1 through $-i, 1, i$ back to -1 .

From (3.15), $z_1 = (1-t)/(1+t)$ and $z_2 = u$. Use the elliptic parametrization of section 6, with modulus k . Then it follows from (6.13) that

$$\alpha_1 = -k \operatorname{sn}^2(v/2), \quad \alpha_2 = k \frac{\operatorname{cn}^2(v/2)}{\operatorname{dn}^2(v/2)}. \quad (\text{A8})$$

and we can verify that

$$\tau_2 = \tau_1 \frac{1-k}{1+k}. \quad (\text{A9})$$

If we define r so that

$$\chi^{-1} = k \operatorname{sn}^2 r \quad (\text{A10})$$

then

$$U^{1/2} = \frac{i k \operatorname{sn} r \operatorname{cn} r}{\operatorname{dn} r}. \quad (\text{A11})$$

As r moves along the horizontal line from $iK'/2$ to $2K + iK'/2$ in the complex plane, χ and $U^{-1/2}$ both go anti-clockwise once round the unit circle, from -1 through $-i, 1, i$ back to -1 , and ω goes upwards along the imaginary axis from $-i\infty$ to $+i\infty$.

We can verify, using (A8) and (A10) and methods similar to those used to obtain (6.14) herein, that

$$\begin{aligned} \frac{1 - U^{1/2}}{1 + U^{1/2}} &= \frac{-i \operatorname{cn}(r - iK'/2)}{\operatorname{sn}(r - iK'/2) \operatorname{dn}(r - iK'/2)} = \rho, \\ \frac{1 - \omega}{1 + \omega} &= k \operatorname{sn}(r - v/2) \operatorname{sn}(r + v/2) = \lambda^{-1}, \end{aligned} \quad (\text{A12})$$

where ρ, λ are the ρ, λ defined as functions of r in (5.1) and (6.5).

This almost completes the identification of our result for f_s with that of McCoy and Wu. Using the definitions in (6.23):

$$A_1(r) = \log \lambda, \quad B_2(r) = i \frac{d}{dr} \log \rho,$$

we see that

$$\frac{d\omega}{1 - \omega^2} = \frac{1}{2} A'_1(r) dr,$$

and we can write (A6) as

$$\mathcal{J} = \frac{i}{4\pi} \int_{iK'/2 - i\epsilon}^{2K + iK'/2 - i\epsilon} A'_1(r) \log \rho(r) dr, \quad (\text{A13})$$

where ϵ is small and positive real (this corresponds to the shift of the ω -integration to just to the right of the imaginary axis).

The integrand is a periodic function of r , of period $2K$, so we can shift the integration down to the real axis, but have to allow for the simple pole in $A'_1(r)$ at $r = v/2$, where, using (6.11), (6.12) and (3.15),

$$\rho(r) = \rho(v/2) = \frac{i \operatorname{cn}(\overline{v}/2)}{\operatorname{sn}(\overline{v}/2) \operatorname{dn}(\overline{v}/2)} = u^{-1} = \coth H'. \quad (\text{A14})$$

Also shifting the contour back a distance K , we get therefore

$$\mathcal{J} = \frac{1}{2} \log \tanh H' + \frac{i}{4\pi} \int_{-K}^K A'_1(r) \log \rho(r) dr. \quad (\text{A15})$$

Substituting this result into (A5) and integrating by parts (using the fact that both $\log \lambda$ and $\log \rho$ are now periodic of period $2K$), we find that McCoy and Wu's result (A2) is equivalent to

$$-\beta f_s = -H' + \frac{1}{4\pi} \int_{-K}^K A_1(r) B_2(r) dr, \quad (\text{A16})$$

and this is indeed the same as our result (6.25).

B. The determinants of Y and C .

Throughout this and the following appendix, k is an integer suffix, *not* the elliptic modulus introduced in (5.15).

B.1. The determinant of Y

To calculate the determinant of the $2N$ by $2N$ matrix Y , we first note that it can be transformed to a matrix composed of two N by N diagonal blocks:

$$\widehat{Y} = \frac{1}{2} [\mathbf{1} + (-1)^N i S] Y \mathcal{B} \quad (\text{B1})$$

where S is defined by (3.35) and \mathcal{B} is the matrix that puts all the odd columns into positions $1, 2, \dots, N$, followed by all the even columns. Thus

$$\mathcal{B}_{k',k} = 1 \text{ if } k' = m(k), \quad \mathcal{B}_{k',k} = 0 \text{ else,}$$

where

$$m(k) = 2k - 1 \text{ if } k \leq N, \quad m(k) = 2k - 2N \text{ if } k > N.$$

From (3.30), it follows that, for $j = 1, 2, \dots, N$ and $1 \leq k \leq 2N$,

$$\widehat{Y}_{j,k} = \frac{1}{2} \rho_k [z_m^{j-1} - z_m^{2N+1-j} + u(-1)^N (z_m^{N+j-1} - z_m^{N+1-j})], \quad (\text{B2})$$

$$\widehat{Y}_{j+N,k} = \frac{1}{2} i \rho'_k [u z_m^j - u z_m^{2N-j} - (-1)^N (z_m^{N+j} - z_m^{N-j})], \quad (\text{B3})$$

where $m = m(k)$, $\rho_m = 1 - (-1)^N \alpha_m$, $\rho'_m = 1 + (-1)^N \alpha_m$.

Using (3.23), it follows that $\rho_k = 1 - (-1)^{m(k)}$, so is zero for $k > N$. Similarly, $\rho'_k = 0$ if $k \leq N$. Hence \widehat{Y} is a block-diagonal matrix, with two N by N blocks.

First consider the top-left block Y_1 , with elements given by (B2) for $1 \leq j, k \leq N$ and $\rho_k = 2$. For small values of N we find (using Mathematica), for arbitrary z_1, z_2, \dots, z_{2N} , that

$$\det Y_1 = c_1 \prod_{j=1}^N (1 - z_{2j-1}^2) \prod_{1 \leq j < k \leq N} (z_{2j-1} - z_{2k-1})(1 - z_{2j-1} z_{2k-1}), \quad (\text{B4})$$

where

$$\begin{aligned} c_1 &= (u-1)(u^2-1)^{N/2-1} \text{ if } N \text{ is even,} \\ &= (u^2-1)^{(N-1)/2} \text{ if } N \text{ is odd.} \end{aligned} \quad (\text{B5})$$

We can prove that this is correct for all N . Consider the first column of Y_1 . Its elements are all polynomials in z_1 , of degree at most $2N$, and z_1 occurs only in this column. The determinant must therefore be a polynomial of this degree in z_1 . If, for $k > 1$, $z_1 = z_{2k-1}$, then columns 1 and k are identical and the determinant vanishes, so $z_1 - z_{2k-1}$ is a factor of this polynomial.

Also, inverting z_1 merely divides all the elements of the column 1 by $-z_1^{2N}$, so $1 - z_1 z_{2k-1}$ must also be a factor.

If $z_1 = \pm 1$, all elements of the first column vanish, so $1 - z_1^2$ must be a factor.

Hence the determinant must be of the form

$$\tilde{c}_1 (1 - z_1^2) \prod_{k=2}^N (z_1 - z_{2k-1})(1 - z_1 z_{2k-1})$$

where \tilde{c}_1 is a polynomial in z_1 . But the other terms have total degree $2N$, so \tilde{c}_1 must be independent of z_1 .

Repeating this argument for the other columns, the determinant must be of the form (B4), where c_1 is independent of $z_1, z_3, \dots, z_{2N-1}$.

To calculate c_1 , left-multiply Y_1 by a lower-triangular matrix \mathcal{M} that ensures the exponents of the $Y_{1,j,k}$ monotonically decrease as N increases. This does not change the determinant.

If N is odd, we take $\mathcal{M}_{j,j} = 1$, and $\mathcal{M}_{j,N=2-j} = -u$ provided $2j < N-2$, else $\mathcal{M}_{j,k} = 0$. Then the terms of highest power in z_{2k-1} in column k are

$$-z^{2N}, -z^{2N-1}, \dots, -z^{(3N+1)/2}, (u^2-1)z^{(3N-1)/2}, \dots, (u^2-1)z^{N+1} \quad (\text{B6})$$

The term in the determinant that is proportional to $z_1^{2N} z_3^{2N-1} \dots z_{2N-1}^{N+1}$ is the product of the diagonal elements of $\mathcal{M}Y_1$ and from our last result this has coefficient

$$(-1)^{(N+1)/2} (u^2-1)^{(N-1)/2}.$$

The corresponding term on the RHS of (B4) is $(-1)^{(N+1)/2} c_1$. They must be equal, so $c_1 = (u^2-1)^{(N-1)/2}$, which is the formula given in (B5) for N odd.

The proof for N even is similar, the main difference being that the term in position $N/2 + 1$ in the sequence (B6) is $(u-1)z^{3N/2}$.

From (B2), for $j, k = 1, \dots, N$,

$$\hat{Y}_{2N+1-j,k+N} = i(-1)^N \{\hat{Y}_{j,k}\} \quad (\text{B7})$$

where $\{\widehat{Y}_{j,k}\}$ is the RHS of (B3), but with u, z_{2k-1} replaced by $-u, z_{2k}$. It follows that if Y_2 is the lower-right block of \widehat{Y} , then

$$\det Y_2 = c_2 \prod_{j=1}^N (1 - z_{2j}^2) \prod_{1 \leq j < k \leq N} (z_{2j} - z_{2k})(1 - z_{2j} z_{2k}), \quad (\text{B8})$$

where

$$\begin{aligned} c_2 &= -(u+1)(u^2-1)^{N/2-1} \text{ if } N \text{ is even,} \\ &= -i(u^2-1)^{(N-1)/2} \text{ if } N \text{ is odd.} \end{aligned} \quad (\text{B9})$$

Multiplying (B4) and (B8) together, and noting from (B1) that $\det \widehat{Y} = (-1)^{N(N-1)/2} 2^{-N} \det Y$, we obtain

$$\det Y = (-2i)^N (1-u^2)^{N-1} \prod_{j=1}^{2N} (1 - z_j^2) \prod_{1 \leq j < k \leq 2N}^{**} (z_j - z_k)(1 - z_j z_k), \quad (\text{B10})$$

where the $**$ on the second product means that it is over all pairs j, k of the same parity (both even or both odd).

Now we use the fact that $z_{2N+1-k} = 1/z_k$ to reduce this result to the form

$$\det Y = 2^N \epsilon_N (1-u^2)^{N-1} \prod_{j=1}^N \frac{(1 - z_j^2)^2}{z_j^{2N}} \prod_{k=1}^{j-1} (z_j - z_k)^2 (1 - z_j z_k)^2. \quad (\text{B11})$$

B.2. The determinant of C

From (3.43) and (4.1), C is the N by N matrix with elements

$$C_{jk} = z_k^{j-1} + (-1)^{N-k} z_k^{N-j}. \quad (\text{B12})$$

For small values of N we find that

$$\det C = \tilde{c} \prod_{1 \leq j < k \leq N}^{**} (z_j - z_k)(1 - z_j z_k), \quad (\text{B13})$$

where again the $**$ means that the product is restricted to j, k such that $j - k$ is even, and

$$\begin{aligned} \tilde{c} &= 2^{N/2} \prod_{j=1}^{N/2} (1 - z_{2j-1})(1 + z_{2j}) \text{ if } N \text{ is even} \\ &= 2^{(N+1)/2} \prod_{j=1}^{(N-1)/2} (z_{2j}^2 - 1) \text{ if } N \text{ is odd.} \end{aligned} \quad (\text{B14})$$

We prove that (B13) is correct for all N in a similar way to our proof above of (B4). The determinant of C is a polynomial in any z_j , of degree $N - 1$. If, for two j, k of the same parity, either $z_j = z_k$ or $z_j = 1/z_k$, then columns j, k of the matrix C will be proportional to one another, so the determinant of C will vanish. Hence $\det C$ contains the product in (B13) as a factor. This product

is a polynomial if each z_j of degree $N - 2$ if N is even. if N is odd, it is of degree $N - 1$ in z_k for k odd, of degree $N - 3$ in z_k for k even.

If $n - k$ is odd and $z_k = 1$, then $C_{jk} = 0$ for all j , so again $\det C$ vanishes. Similarly if k is even and $z_k = -1$. It follows that \tilde{c} must contain the products in (B14) as factors. The combined products in (B13), (B14) are of degree $N - 1$ in each factor, so (B14) is correct to within multiplication by a constant.

To obtain this constant, first look at the contribution to $\det C$ of lowest degree in z_{N-1} and z_N . This is of degree zero in both, so is obtained by setting $z_{N-1} = z_N = 0$. The last two columns of C then have non-zero entries only in rows 1 and N :

$$C = \begin{pmatrix} .. & .. & .. & 1 & 1 \\ \# & \# & \# & 0 & 0 \\ \# & \# & \# & .. & .. \\ \# & \# & \# & 0 & 0 \\ .. & .. & .. & -1 & 1 \end{pmatrix} \quad (\text{B15})$$

Hence

$$\det C = 2(-1)^N \times \overline{C} \quad (\text{B16})$$

where \overline{C} is the determinant of the $N - 2$ by $N - 2$ matrix denoted by the $\#$ elements in (B15). Hence $\overline{C}_{j,k} = C_{j+1,k}$, $1 \leq j, k \leq N - 2$. From (B12), it follows that \overline{C} is the same as C , with N replaced by $N - 2$ and all elements multiplied by $z_1 z_2 \cdots z_{N-2}$, i.e writing C as C_N ,

$$\det C_N = 2(-1)^N z_1 z_2 \cdots z_{N-2} C_{N-2}. \quad (\text{B17})$$

Iterating, it follows that the term of lowest degree in z_N, z_{N-1} , then in z_{N-2}, z_{N-3} , etc. of $\det C$ is:

$$2^{N/2} (z_1 z_2)^{(N-2)/2} (z_3 z_4)^{(N-4)/2} \cdots (z_{N-3} z_{N-2})$$

if N is even, and

$$-(-2)^{(N+1)/2} z_1^{(N-1)/2} (z_2 z_3)^{(N-3)/2} (z_4 z_5)^{(N-5)/2} \cdots (z_{N-3} z_{N-2})$$

if N is odd.

On the other hand, the corresponding coefficient of the term of lowest degree in the combined products in (B13), (B14) is 1 except when $N = 3 \pmod{4}$, when it is -1. This minus sign cancels the minus signs in the last equation, leaving the remaining coefficients as $2^{N/2}$ if N is even, $2^{(N+1)/2}$ if N is odd. Thus \tilde{c} is as given by (B14) and we have proved the identity (B13), (B14) for $\det C$.

C. Calculation of E .

First look at the factor $\phi\phi'/\eta$ in (4.10). From (4.6), (4.7),

$$\phi\phi' = \zeta^{-2N} \prod_{k=1}^N [(1 - uz_k)^2 - z_k^{2N} (u - z_k)^2]. \quad (\text{C1})$$

Each z_k is a zero of the polynomial $P(z)$ of (3.21), so

$$z_k^{2N} = \frac{(1 - tuz_k)(1 - uz_k/t)}{(z_k - tu)(z_k - u/t)}. \quad (\text{C2})$$

Using the square root of this equation, we can write (C1) as

$$\phi\phi' = (-1)^n \prod_{k=1}^N \frac{(1-t)^2(1-u^2)\zeta^{-1}(1-z_k^2)}{[(tu-z_k)(tu-1/z_k)(t/u-z_k)(t/u-1/z_k)]^{1/2}} \quad (\text{C3})$$

From (3.21),

$$P(z) = (z^2 - 1) \prod_{k=1}^N (z - z_k)(z - 1/z_k),$$

so it is exactly true that

$$\prod_{k=1}^N (tu - z_k)(tu - 1/z_k) = 1 - u^2$$

$$\prod_{k=1}^N (t/u - z_k)(t/u - 1/z_k) = (1 - u^2) (t/u)^{2N}$$

and hence, using (4.8), that

$$\frac{\phi\phi'}{\eta} = \epsilon_N i^{-N} (1-t)^{2N} (1-u^2)^{N-1} \left(\frac{u}{t\zeta}\right)^N \prod_{k=1}^N (1-z_k)^2 \quad (\text{C4})$$

We now look at the expression $\det Y / (\det C)^2$. We have to consider separately the cases N even and N odd.

C.1. N even

First we focus on the case when N is even and define

$$n = N/2,$$

then we can write (4.14) as

$$\frac{\det Y}{(\det C)^2} = \frac{\mathcal{L}(1-u^2)^{N-1}}{\zeta^{2N}} \prod_{j,k=1}^n (z_{2j-1} - z_{2k})^2 (1 - z_{2j-1} z_{2k})^2 \quad (\text{C5})$$

Define polynomials $P_1(z), P_2(z)$, of degree $N+2, N$, respectively, by

$$P_1(z) = (z^2 - 1) \prod_{j=1}^n (z - z_{2j})(z - 1/z_{2j}),$$

$$P_2(z) = \prod_{j=1}^n (z - z_{2j-1})(z - 1/z_{2j-1}). \quad (\text{C6})$$

Then

$$\frac{\det Y}{(\det C)^2} = \frac{(1-u^2)^{N-1}}{\zeta^{2N}} \prod_{j=1}^n \frac{z_{2j}^N (z_{2j}-1)^2 P_1(z_{2j-1})^2}{(z_{2j-1}-1)^2} \quad (C7)$$

and from (3.21),

$$P_1(z) P_2(z) = P(z). \quad (C8)$$

Consider the functions

$$J_{\pm}(z) = z^N \left[\frac{1-ut/z}{1-t/uz} \right]^{1/2} \pm \left[\frac{1-utz}{1-tz/u} \right]^{1/2} \quad (C9)$$

taking both square roots to be analytic functions of z in an annulus containing the unit circle, positive real when z is real and positive and $t/u < z < u/t$. This line segment contains the zeros $z_N, 1, z_{N+1}$ of $P(z)$.

From (3.21), the zeros of $J_{\pm}(z)$ are also zeros of $P(z)$. By considering the limit when $t \rightarrow \infty$, we can see that the zeros of $J_+(z)$ are the z_j for j odd, i.e. z_1, z_3, \dots, z_{N-1} and $1/z_1, 1/z_3, \dots, 1/z_{N-1}$, and these are the zeros with $\alpha_j = -1$.

Thus $J_+(z)$ and $P_2(z)$ have precisely the same zeros. Further, if we take $|z|$ to be of order 1 and expand the RHS of (C9) in powers of t , then to order t^N , $J_+(z)$ is a polynomial in z of degree N , with leading term z^N . We therefore expect that if $|z|$ is of order one, then if we neglect terms of order $(t/u)^N$,

$$P_2(z) = J_+(z). \quad (C10)$$

Since $J_+(z) J_-(z) = P(z)/[(1-uz/t)(1-tz/u)]$, it also follows that

$$P_1(z) = (1-uz/t)(1-tz/u)J_-(z). \quad (C11)$$

Let $z = z_{2j-1}$, then the two terms on the RHS of (C9) for $J_-(z)$ are equal; taking their geometric mean, we obtain

$$P_1(z)^2 = -\frac{4z^{N+2}u^2}{t^2} (t/u - z)^{3/2} (t/u - 1/z)^{3/2} (ut - z)^{1/2} (ut - 1/z)^{1/2} \quad (C12)$$

Substituting this expression into (C7), we obtain

$$\frac{\det Y}{(\det C)^2} = \frac{(2iu)^N (1-u^2)^{N-1}}{t^N \zeta^N} P_2(t/u)^{3/2} P_2(tu)^{1/2} \prod_{j=1}^n \frac{z_{2j-1}^2 (1-z_{2j})^2}{(1-z_{2j-1})^2} \quad (C13)$$

If $|z| < 1$, it is still true that (C10) holds to within relative errors of order $(t/u)^N$, provided we only retain the second term in (C9). Hence in (C13) we can take

$$P_2(t/u) = \left[\frac{1-t^2}{1-t^2/u^2} \right]^{1/2}, \quad P_2(tu) = \left[\frac{1-u^2 t^2}{1-t^2} \right]^{1/2}. \quad (C14)$$

Now we take the ratio of (C4) to (C13). Using the definition (4.10), we obtain

$$E = \frac{(1-t)^{2N} \tilde{e}^2}{2^N P_2(t/u)^{3/2} P_2(tu)^{1/2}} \quad (C15)$$

where

$$\tilde{e} = \prod_{j=1}^n (1 - z_{2j-1})(1 - 1/z_{2j-1}) = P_2(1). \quad (\text{C16})$$

From (C9), (C10),

$$P_2(1) = 2 \left(\frac{1 - ut}{1 - t/u} \right)^{1/2},$$

so

$$E = \frac{(1-t)^{2N} (1-ut) (1-t^2/u^2)^{3/4}}{2^{N-2} (1-t/u) (1-u^2 t^2)^{1/4} (1-t^2)^{1/2}}. \quad (\text{C17})$$

C.2. N odd

Taking N to be odd, define the integer

$$n = (N - 1)/2 \quad (\text{C18})$$

and the polynomials

$$\begin{aligned} P_1(z) &= (z - 1) \prod_{j=1}^{n+1} (z - z_{2j-1})(z - 1/z_{2j-1}), \\ P_2(z) &= (z + 1) \prod_{j=1}^n (z - z_{2j})(z - 1/z_{2j}). \end{aligned} \quad (\text{C19})$$

Like the P_1, P_2 of the N even case, they are of degree $N+2$ and N , respectively. Eqn. (C8) is still true and (C10), (C11) still hold to within terms of relative order $(t/u)^N$ when $|z| = 1$.

Instead of (C7), we have

$$\frac{\det Y}{(\det C)^2} = \frac{i(1-u^2)^{N-1}}{2\zeta^{2N}} \prod_{j=1}^{n+1} z_{2j-1}^{N-1} (1 - z_{2j-1}^2)^2 \prod_{j=1}^n \frac{P_1(z_{2j})^2}{(1 - z_{2j})^2} \quad (\text{C20})$$

Since z_2, z_4, \dots, z_{2n} all lie on the unit circle, we can replace $P_1(z)$ in this equation by the RHS of (C11). Again the two terms on the RHS of (C9) are equal, and equation (C12) still applies. Using (C19), equation (C20) becomes

$$\frac{\det Y}{(\det C)^2} = \frac{(-1)^n i 2^{N-2} s (u - u^3)^{N-1} P_2(t/u)^{3/2} P_2(tu)^{1/2}}{t^{2n} \zeta^N (1 + t/u)^{3/2} (1 + tu)^{1/2}} \prod_{j=1}^n \frac{z_{2j}^2}{(1 - z_{2j})^2} \quad (\text{C21})$$

where

$$s = \prod_{j=1}^{n+1} \frac{(1 - z_{2j-1}^2)^2}{z_{2j-1}}.$$

From (C19), (C10) and (C11),

$$\prod_{j=1}^{n+1} \frac{(1 + z_{2j-1})^2}{z_{2j-1}} = -\frac{P_1(-1)}{2} = \frac{u(1 + t/u)^{3/2} (1 + ut)^{1/2}}{t}$$

$$\prod_{j=1}^n \frac{(1 - z_{2j})^4}{z_{2j}^2} = \frac{P_2(1)^2}{4} = \frac{1 - ut}{1 - t/u},$$

Using these relations and (C14), taking the ratio of (C4) to (C21), we obtain

$$E = \frac{(1 - t)^{2N} (1 - ut) (1 - t^2/u^2)^{3/4}}{2^{N-2} (1 - t/u) (1 - u^2 t^2)^{1/4} (1 - t^2)^{1/2}}, \quad (\text{C22})$$

which is the same as the N even result C17, even though there are significant differences between their calculations.

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